

NEW SYSTEM
OF
ARITHMETICK,
Theoretical and Practical.

WHEREIN

The SCIENCE of NUMBERS
IS DEMONSTRATED

In a REGULAR COURSE from its FIRST PRINCIPLES,
thro' all the PARTS and BRANCHES thereof;

Either known to the ANCIENTS, or owing to the
Improvements of the MODERNS.

The PRACTICE and APPLICATION to the *Affairs of Life*
and *Commerce* being also Fully Explained:

So as to make the Whole a

COMPLETE SYSTEM of THEORY,

For the Purposes of MEN of SCIENCE;

And of PRACTICE, for MEN of BUSINESS.

By *ALEXANDER MALCOLM*, A. M.
Teacher of the MATHEMATICKS at *Aberdeen*.

L O N D O N:

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M.DCC.XXX.

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T O

THE RIGHT HONOURABLE

William Cruikshank, *Esq;*

L O R D P R O V O S T :

James Moorison,

John Gordon,

William Mowat,

Hugh Hay,

Baillies :

Alexander Livingston, *Dean of Guild;*

Alexander Robertson, *Treasurer :*

And the Remanent Members of the Town-Council of

A B E R D E E N.

My L O R D, &c.

THE Subject and Design of the following Work, with
My Relation to the Town of *Aberdeen* as a publick
Teacher, naturally directed me to its Governours
for Patronage. As the Encouragement I have already met
with from the Town, and in particular from its present
Magistrates

Magistrates and Council, both with respect to this Work, and to my publick Employment, is but the Native Consequence of that Noble Disposition for promoting Learning and all good Education, which is the well-known Character of *Aberdeen*; so, I believe, I should rather offend than please by attempting any Apology for this Address; or running into the Common Way of flattering Dedications. I know how disagreeable it is to Generous Minds, to hear their own just Praises: But I hope you will forgive me, if I avoid the Appearance of Ingratitude, by making this publick Acknowledgement of the Kindness and Civility with which You have used,

My LORD, &c.

Your Lordship's, &c. Most

Obedient Humble Servant,

ALEX^R MALCOLM.

P R E F A C E.

WHEN a Subject has gone thro' so many Hands as Arithmetick has done, a new Book cannot want many Prejudices against it: and therefore to send it into the World without some introductory Account of it, is no better than laying it down at random; or, more properly, exposing it. 'Tis an unreasonable Neglect of something, that equally concerns the Author and the World: for if an Author has endeavour'd to do something more useful and complete upon any Subject than has been already done, and thinks he has in some measure succeeded; as the telling the World so, may be done without any Breach of Modesty, so it appears to me equally just and necessary to explain particularly wherein the Improvements and Advantages of the Work lie; that every one may see how far it answers their Purpose, and deserves their Encouragement. It must stand upon its own Basis, no doubt; yet nothing seems more honest and reasonable than this kind of Invitation to look into it. It may be objected, I know, that here is only the Author's Word for this Account, which is a partial Testimony: But if it be consider'd, that he ventures his Credit as well as the Success of his Work, upon a fair Representation, this, I may reasonably hope, will incline the more Candid and Charitable, to believe that it is so. And upon this Hope I presume to give you the following Account of this Work.

ARITHMETICK is a Subject of that Extent, that in some Respects it can never be exhausted; and of that Value, as to deserve all the Study and Pains that can be bestow'd upon it. It is certain, there is no End in the Knowledge of Numbers; but as to a just and rational System of the Science, one would think that can't be a thing still wanted, after so many Books already written on this Subject: Nevertheless, in my Opinion, we are far from having any such thing, in our Language at least; and as to what may be in other Languages, I can only say, That I have not found it in the Books that have come to my hands.

But that I may express my Sentiments upon this Matter a little more particularly, as necessary to introduce an Account of the present Work, I shall first observe, That Arithmetick is to be consider'd in two Respects, *viz.* either in its *Theory*, which contains the Abstract and Speculative Knowledge of pure Numbers; or in its *Practice*, which contains the Application of that Theory to human Affairs. The Theory is first in order of Science; the Application supposing and depending upon it: So that there can be no Application without some previous abstract Knowledge of Numbers; that abstract Knowledge being the very thing to be apply'd. But then it is to be consider'd, that there is a great Difference betwixt understanding the Sense and Meaning of any Proposition, or of any Rule in Arithmetick, so as to be able to follow its Directions; and knowing the Reason and Demonstration of the Truth of that Proposition or Rule. Hence it is, that there are two very different ways of studying and knowing Arithmetick. The Generality who practise Arithmetick, and even many whose Business requires a Knowledge much above the more common Parts, yet understand little or nothing of the Reason or Demonstration
of

of what they do, because they study not the Theory of Arithmetick, and ask no more than plain Rules for the Practice, so far as they have use for it. But others, considering Arithmetick as a Science founded upon Principles and Reason, require a Demonstration for every thing.

Answerable to those two different Demands, the Books of Arithmetick which we have, are of two kinds; the *Practical* and *Theoretical*. The practical Books are most of them small Treatises of the first and more simple Elements and Applications of Arithmetick: But besides that they go a very short way into the Science, they have also left us without the least Reason for any thing they deliver, more than what is in some Cases evident from the Nature of the thing: Taking all the rest for granted, or leaving the Demonstration to the Theorists.

The Theoretical Writers have treated Arithmetick as a Science, by demonstrating what they deliver: Some of them treat the Subject altogether abstractly, without any particular Application, as *Parsons* in his *Clavis Arithmeticae*: others with the Theory join also the Application, doing more or less in it as they have thought fit, as *Ward*, *Tacquet*, and others. Of this Class, again, some begin in the natural Order with the simple Elements: Others omit these, supposing them already understood, and fall in at once into a more advanced Theory. Such Elements of Arithmetick we have in *Euclid's* 7th, 8th, and 9th Books; and this Method has been imitated both anciently and of late. *Tacquet* has given us those three Books of *Euclid* for the Elements of Arithmetick; placing them before what he calls the practical Arithmetick, which contains the common Principles and Rules, and some things relating to Progressions, with the Extraction of the Square and Cube Roots; all very nearly explained and demonstrated, as far as he carries the matter, [excepting one small Mistake I have occasionally taken notice of in the following Work; and his Demonstrations of the Square and Cube Roots, which appear to me deficient.] But I could never understand the Reason of this Order; he could not certainly mean that those Elements of *Euclid* were to be studied before the more simple Elements, which without doubt *Euclid* supposed as necessarily previous to his.

But of all the Works of this Class, I have found none which I can reckon a plain, rational and compleat System or Institution of the Science of Arithmetick; either from the want of several things, even elementary and fundamental in the Science, (which is a common Fault with them all) or being too concise and short in other things; or from some other Difficulty or Fault in the Method; owing, perhaps, to their particular Views and Designs; but which answers not to my Idea of the thing wanted. How unaccountable (for Example) is it in Mathematical Writers, to leave several things undemonstrated, to send us to *Euclid* for others, or give us but very general and imperfect Hints of a Demonstration? But I have done; for to be more particular, would not only be useless, but perhaps be misconstrued to a worse Sense.

FROM this general Account of Arithmetick, and the different Ways of treating it, the Thing wanted will easily appear to be this, *viz.* A Treatise, wherein the Science is deduced from its first Principles; and carried on with clear and accurate Demonstration thro' all the fundamental Branches of its Theory and Practice, with the more considerable Improvements hitherto made in the Science; all disposed according to the most easy and natural Connection and Dependence of the several Parts; hereby uniting the whole into one regular and complete System. Again, in such a System Numbers must not only be consider'd abstractly, or purely as Numbers, but we must also consider their Application to particular Subjects, that we may have a compleat Course of what we call the practical Arithmetick; which, besides the more simple Elements of Practice, or fundamental Rules of Operation with pure and abstract Numbers, explains the Application of those Rules to the more common and ordinary Subjects of human Affairs.

Such

Such a complete and rational System of Arithmetick, accommodated to the Purposes both of the practical and speculative People, I have endeavour'd to give in the following Work; of the Contents and Order of which, I shall give you a more particular Account immediately: But before this, it will be proper to make the following general Reflection upon the System of Arithmetick, both as to its Theory and Practice; which is this:

ARITHMETICK taken abstractly, or in its Theory, being the first great Branch of the Mathematicks, its Application is to be found not only in the common Affairs of Life and Society, but in all the Sciences that are call'd Mathematical, (which have all their different Uses in Society.) But then observe, that it is not to be expected that a Course of Arithmetick should explain such Applications as require the Knowledge of other Sciences; for then we should be obliged to bring into it all the Mathematical Sciences; since to understand its Application to the Subjects of these Sciences, does necessarily require our understanding the Principles of them. For Example, if 'tis propos'd to find what Part or Parts any lesser Sphere is of another, the Lengths of their Diameters being known? This is a Question solvable by Arithmetick; yet the Reason of the Rule goes farther than Arithmetick, for it depends upon Geometry, *viz.* upon that Geometrical Truth, that Spheres are to one another in Proportion as the Cubes of their Diameters; and so belongs to Arithmetick only as this is applicable in Geometry, and supposes the Knowledge of this Science.

From this it is evident, that the Applications proper to be explained in a System of Arithmetick, are only such as relate to the more ordinary Affairs of Life, which require the previous Knowledge of no other particular Science, and depend immediately and directly upon the Consideration of the Numbers of things, and some other common Circumstances. Such are all the simple Applications of the common fundamental Operations of Arithmetick, either in Whole Numbers or in Fractions; and the Applications of the general Rules of Proportion in the common Subjects of Trade and Commerce: For in all this there is no more requir'd, but a careful Attention to the Sense of the Question, and the true Effect of the Rules of Arithmetick.

Again it is to be observ'd, That as the Theory of Arithmetick is an abstract Science, independent of all those Subjects to which it may be apply'd, it is therefore necessary that we have a complete System of the Theory of Numbers, consider'd purely and abstractly by themselves; this being presuppos'd in the Solution of all Questions in other Sciences, which have any Dependence upon Numbers. The next thing I observe here, is, That tho' there be many Truths discovered in the Theory of Arithmetick, of which there has been no Use or Application yet found, this is no reason why those things should be neglected or kept out of the System; they are still a Part of the Science, which we ought to enlarge more and more, as far as we can: One Age may find the Use of the Theory which a former has invented; as undoubtedly has been the Case, with respect to most part of the Theory both of Arithmetick and Geometry. I shall but add this one thing more, *viz.* That tho' many things in the Science of Numbers were suppos'd to be of no particular Use in human Affairs, yet as the Mind of Man is made for Knowledge and Contemplation, and the Pleasure arising from the Perception of Beauty and Order in other things, is allow'd to be worthy of rational Natures; the Contemplation of the surprising Connections, the beautiful Order and Harmony of Relations and Dependencies found among Numbers, is not less reasonable: And if to this be join'd the vast Extent of the Use and Application of Arithmetick, the Reasonableness and Necessity of explaining the Theory of Numbers so largely as I have done, will easily be allow'd.

I should now come to the Contents of the following Work, but as some particular Circumstances oblige me to take notice of two late and well-known Authors, Mr. Hill and Mr. Hatton, I shall first discuss what I think necessary to say as to their Works.

Mr. Hill's Book, which he calls *Arithmetick in Theory and Practice*, is remarkable chiefly for the very uncommon Recommendation it carries with it from a very considerable Master. We are told by Mr. Ditton, That take this Author purely as an Arithmetician,

metician, he has not only done more and much better than *Wingate, Cocker, Leyburn*, or any other of the Writers in our Tongue, but indeed all that can be done by Arithmetick; and therefore (says he) if no other Book on this Subject comes out till this Performance is really mended, I'm satisfy'd we shall have no new Book of Arithmetick very soon.

Now here was such a Defiance, and from such a Hand, that they were bold enough who ventur'd first to write after it, and even without the least Apology, or Notice taken of this Challenge, as several have done: Whatever Reason others thought they had for such a Conduct, I thought it necessary for my own Vindication to make the following Remarks on this Book and its Recommendation.

That Mr. *Hill* has several things that are not common, I do acknowledge; but for his having done much better than all that went before him, 'tis not my Business to determine; what I'm concern'd in chiefly, is the Assertion of his having done all that can be done by Arithmetick, a thing I was much surpriz'd to hear from so good a Judge as Mr. *Ditton*; and because if this be true, then what I offer to the World must be either impertinent or superfluous; it can't be thought out of my Road if I enquire a little into the Truth of this Assertion. By the mention Mr. *Ditton* makes of *Algebra*, it appears to me, that he would have nothing admitted into Arithmetick that is any way owing to *Algebra*. Now, supposing this were reasonable, yet the Book in question will still be found both very defective in what it ought to contain to answer so great a Character, and also to have many things that belong not to a pure Treatise of Arithmetick. In the first place, with what Truth and Justice can it be said, That a Book contains all that belongs to Arithmetick, and that one needs learn no more, (as he also says) which, (besides that there is no Demonstration, and consequently no Science) wants many things that are fundamental and necessary, and yet do not absolutely depend upon *Algebra*; (tho' they may be made easier in many things by its help) particularly in the Doctrine of Proportion: For tho' we have here several Propositions relating to this Subject, yet we are very far from having any thing like a just and orderly Treatise of Proportion: Nor (to mention no more of its Defects) have we any of the rest of the fundamental and curious Theory of Numbers, contain'd in *Euclid's* 7th, 8th, and 9th Books. Again, if we must exclude what is any way owing to *Algebra*, then most of what is uncommon in Mr. *Hill*, as upon *Progressions, Interest, Logarithms, Combinations, and Extraction of Roots*, do not belong to Arithmetick: And if these belong to *Arithmetick*, notwithstanding their Dependence upon *Algebra*, then so must a great many other things not to be found in Mr. *Hill's* Book. But I have said enough, and shall leave you to judge by the following Work, whether that Book contains all that can be done by *Arithmetick*, and consequently what to think of this extraordinary Recommendation, which indeed is more faulty than the Book itself.

Mr. *Hutton's* Book, which I have here in my view, is his *Intire System of Arithmetick*: If this Book had answer'd the promising Title, my Labour had been prevented; but I could not help judging otherwise of a Book that not only leaves us without Demonstration in most things, and sometimes gives us a mere Proof of a particular Example, instead of a general Demonstration; but which, in a word, comes very far short both of the Contents and Order due to an *Intire System*. As I'm no further concern'd in the Criticism of another Man's Work, than it is necessary to vindicate my own, I shall be content with this general Reflection upon this Work, and leave it to an impartial Comparison to justify what I have alledg'd, and determine whether there was not yet wanting a more *Intire System*. There is one thing more I must say here, *viz.* That as I think it is every one's business who writes upon any Subject, to discover the Errors (especially if they are of any consequence) committed by others, so I hope there will be no Misconstruction made of my Design, in exposing some Errors I have found in this or in any other Author: It is the Treatment I expect my self, and shall receive without complaining.

plaining*. But I'm oblig'd for the sake of the Publick, to observe here, that as Mr. *Hatton* has found fault with Dr. *Harris's* Rule for the discounting of *Simple Interest*, for Money paid before it is due; so he ought in justice to have told the World, That the Tables of Discount in his own *Index to Interest*, printed *Anno* 1711, are calculated by the same false Rule; that no body may be longer impos'd upon by them. And that you may not think the Consequence inconsiderable, take this Example: The Discount of 1000 *l.* paid 90 Days before due, is by his Tables *l.* 13.9571, the Discount being at 6 per Cent. whereas, according to the True Rule, the Discount ought to be *l.* 14.57882

I proceed now to a more particular Account of the following Work, which is divided into six Books.

B O O K I.

In this Book I have largely explain'd and demonstrated the first simple Principles and fundamental Operations of Arithmetick in *Integers* or *Whole Numbers*: In which, after the Principles and Rules for the Management of pure and abstract Numbers, I have separately explain'd the Use and Application of these Rules to particular Subjects, such as occur in human Affairs.

B O O K II.

Here you have fully handled the Doctrine of *Fractions*, where I have first explain'd (in a way I think very easy and demonstrative) the general Nature and Theory of *Fractions*, as a necessary Foundation for understanding the Reason of the Practice; which I have next fully explain'd both in *Vulgar* and *Decimal Fractions*, as they are distinguish'd. Only what we call *Infinite* or *Circulating Decimals*, are refer'd to Book 5. for the sake of the Demonstration.

Observe, As these two Books contain the first and fundamental Principles and Rules of Arithmetick; and as the right understanding of the Foundations of any Science is of great Importance, I have therefore enlarg'd and improv'd every Part almost with such particular Explications and Rules, as will, I hope, be of great use for attaining to a just and perfect Idea of this Science in its Fundamentals, and a more masterly Practice.

B O O K III.

Contains the Doctrine of the *Powers* and *Roots* of Numbers; wherein I have first particularly explain'd, the Nature and Theory of those Numbers call'd *Powers* and *Roots*. After this you have the Rules for raising or forming *Powers*, and *Extracting Roots* in *Integers* and *Fractions*, where I have explain'd Sir *Isaac Newton's* famous Rule call'd the *Binomial Theorem*, and some other curious things relating to the *Extraction* of *Roots*. You have here also what is call'd the Arithmetick of *Surds*, which contains a more particular Application of the preceding Theory to *Roots*, especially those call'd *Surds*. Lastly, you have all the Propositions of the 2^d Book of *Euclid*, which are applicable to *Numbers*, with some others of the same kind.

Observe, As to the Contents of this Book, that excepting the common Rules for extracting the *Square* and *Cube Roots*, all the rest of this curious Branch of Arithmetick is to be found only in our Books of *Algebra*: because the Use of it is above the common Affairs of Life, and occurs chiefly in the higher Applications of the *Algebraick* Art; and also because the Demonstration of it can hardly be made without the help of *Algebra*. But as it is directly and immediately a Part of the Theory of Numbers; which does indeed no otherwise belong to a Treatise of *Algebra*, than any other thing in Arithmetick,

* The Particulars I have censured in Mr. *Hatton's* Book, you'll find in Book 6. Chap. 6. Quest. 6. and in Chap. 11.

which may be demonstrated by the help of the *Algebraick* Method. Also, since I have taken that Method of Demonstration (of which I shall give you a particular Account afterwards) I have therefore given it its due place in the *System of Arithmetick*. I must also observe, that tho' the Writers of *Algebra* have taken this Part of the Theory of *Arithmetick* into their Province, yet it is not, in our Language, treated so fully and particularly as it ought to be; many things being left without Demonstration that to me seem far from being self-evident.

For the Extraction of *Roots*, especially those above the *Square* and *Cube*, there are easier Methods, owing also to the *Algebraick* Art; but as they exceed the Limits prescrib'd to this *System*, they must therefore be sought elsewhere.

B O O K IV.

Contains the *Doctrine of Proportion* in all its Branches, as distinguished into *Arithmetical*, *Geometrical* and *Harmonical*. In each of which, as I have endeavour'd to make the fundamental things clear and plain, so I have omitted nothing worth knowing, in this great and useful part of *Arithmetick*, that I could any where find, or that my own Study could furnish: Whereby, as you have all that our common Books contain, so you have many other things to be found only in such Authors as are not in every body's hands; and many things intirely new, for what I know. And, in both those last two kinds, besides what is mix'd here and there, there are some more considerable Additions; particularly upon *Arithmetical Progressions*, in *Chap. 2. §. 2.* All that is from *Schol. 2.* (after *Probl. 2.*) is intirely new. The *Chapters 5, and 6.* with the *Appendix* to this Book, contain things uncommon, and for the most part altogether new; (see *Contents* more particularly.) So that I dare presume to say, you have here a more compleat System of the Doctrine of Proportions than can be found else where, in our Language at least.

As to the Subject of *Chap. 6.* which is *Harmonical Proportion*, I have this Observation to make, That as Musick in its first Principles depends altogether upon Numbers, so the Knowledge of the Application of Numbers to Musick, which I may call the *Arithmetical Theory* of it, is so very useful and entertaining, that 'tis pity it were so little understood, as I doubt it is, both by the Practisers and Lovers of Musick. What was proper or necessary to be done in this Work, concerning that Application, I have done it; and if any one wants a particular Treatise upon this Subject, they will find it in a Book call'd, *A Treatise of Musick, Speculative, Practical, and Historical*; which is to be found with the Booksellers to whom the present Work belongs.

B O O K V.

This Book is a Miscellany of various things, which are not comprehended under one common Name; and consists of VI. Parts, in as many different Chapters; whose Contents are as follow.

1. The Doctrine of *Prime* and *Composite* Numbers; a fundamental and curious Branch of the Theory of *Arithmetick*.

This is a great Part of the Doctrine of *Euclid's 7th, 8th, and 9th Books of Elements*; which contain, besides, many things relating to the Doctrine of *Proportion*; but those I have put in their due Place with the rest of that Doctrine, which is not so complete in *Euclid* as it has been made since: but as my Method of Demonstration is generally different from his, (tho' in some things there can't be a better than his, and perhaps no other;) so I have not only deliver'd this Theory in a different, and, as I think, a more natural Order; but by means of the *Algebraic* Method, I have gain'd these Advantages, *viz.* That several things are made *Corollaries* to others, which have a sufficiently tedious Demonstration in *Euclid*. Again, several Propositions are made universal, which are limited

mitted in *Euclid* to a few particular Cases: And in others, which can be prov'd only by an Induction of Particulars, I have made the Universality of the Induction more clear and evident, by another Method of Reasoning.

There are here also many things which are not in *Euclid*; Part of which I met with in some rare Books, and others occur'd to my own Study and Observation; particularly the 3^d Section is intirely new. I shall mention but one thing more, that is, a *New* and very Easy Way of finding all the *Prime* and *Composite* Numbers within any given Limit: of which I have given an Example, extended only to 999. The Form of the Table in which they are collected, is much the same with that in Dr. *Pell's* Edition of *Brancker's Algebra*; tho' the Rule by which I have compos'd it be vastly more easy than what's given there.

2. The curious Theory of *Figurate Numbers*; a thing but just touch'd upon in any *English* Book, of my Acquaintance. I have met with more of it in some others, but either without Demonstration, or so much out of my Method, that I could make no use of it. And here the Advantage of the *Algebraick* Method was manifest, by which, several of those things are very simply and easily demonstrated, that otherwise have a very difficult and tedious Demonstration: and without which other things could not, I doubt, be demonstrated at all. To the same means also I owe several things here, that I found in none of my Authors; whereby I have carried this Part further, and, by putting the whole together in a just Order, have given it a more perfect Form than I have any where found it in.

Here you have a new *Canon* for the *Coefficients* of the *Powers* of a *Binomial Root*; and several curious Propositions, relating particularly to Square Numbers: With Rules for summing the Series of the Squares and Cubes of the *natural Progression* of Numbers 1. 2. 3. 4. &c. without actually raising these Powers and adding them together; and a Method of raising *Canons* for summing any of the higher Powers.

As to the Use of this Part, whatever else it may be (which in *Mathematical Affairs* is considerable) you have thro' the whole, remarkable Examples of what I formerly mention'd, *viz.* Of beautiful and surprizing Order and Connection among Numbers.

3. Of *Infinite Series* of Numbers; particularly of decreasing Geometrical *Progressions*, (some useful Applications of which you have in the next Chapter) and of those *Increasing Series*, which are the chief and fundamental things of what the Mathematicians call the *Arithmetick* of *Infinities*; of which they have made a noble Use in *Geometry*; having hereby particularly found many useful practical Rules, for the Mensuration of *Solids* and gauging of Vessels. What I have done here belongs properly to *Arithmetick*. The Application of it to *Geometry* you'll find in *Sturmy's Mathesis Enucleata*, or *Ward's Introduction*. But the whole Doctrine and Application at large, is to be sought from the celebrated Author of it, Dr. *Wallis*.

4. The Theory and Practice of *Infinite* or *Circulating Decimals* (referr'd to this Place for the sake of the Demonstration) which, with what is already done in *Book 2^d, Chap. 2^d*, makes a compleat System of *Decimals*.

Dr. *Wallis* is probably the first, as he has himself observ'd, who has distinctly consider'd this curious Subject of *Circulating Decimals*. He has given us the fundamental Theory of it, but without Demonstration; nor has he meddled with the practical Part, or Way of managing *Infinite Decimals* in Arithmetical Operations. Mr. *Brown*, in his *Decimal Arithmetick*, has handled but one single Case of the Practice, and that not completely neither. Mr. *Cunn* (who is the last Author I know upon this Subject) in his *Treatise of Fractions*, has in his way given us all that Dr. *Wallis* lays upon the Theory, yet without any Demonstration, and a few other obvious things, tending more immediately to the Practical Part; which he has handled at full length, giving us Rules for all Operations and all Cases: But as he demonstrates none of those Rules, (which perhaps he reserv'd for another Work) he has also chosen to express them in such a manner, as to let the

Reason as far out of view as possible, which has this Effect, that in the Rules of Multiplication and Division (which are the more complex and difficult Parts) his Directions are not so easily follow'd; and are besides much harder for the Memory than the Method I have chosen, which depends all upon the easy and natural Explication of one single Proposition, *viz.* The finding the finite Value of (or Vulgar Fraction equal to) any circulating Decimal: for tho' the Demonstrations are omitted, the Rule ought to be as simple and easy as possible. But I must observe this further Effect of Mr. *Cunn's* Way of delivering these Rules, That by themselves one could never, or very hardly, be led into the Reason of them, nor consequently into the way I have chosen; so that it will be the more easily believ'd that the Rules I have given, are the Effect of Speculations made upon this Subject, before I saw this Book; which I mention for this Reason only, that I may not be thought ungrateful to one whom I acknowledge the first Author upon this Practice, from whom therefore I might otherwise be supposed to have borrowed or deduced all that I say; and yet I do acknowledge I owe him one or two useful Hints. I have only one thing more to add, *viz.* That his Rule for the Addition of *Circulates* having compound *Repetends* is insufficient for a general Rule; it will bring out the true Answer in some Cases, but is not universally good for all Cases: the comparing it with the Rule I have here demonstrated will shew the Difference, and the Truth of what I say.

5. The *Logarithmick Arithmetick*; wherein the Nature, Construction, and Use of those admirable Numbers call'd LOGARITHMS are explain'd and demonstrated.

The Manner of constructing or making Logarithms, which I have explained here, is that of the Noble Inventor, the Lord *Neper*, because its Demonstration is more simple and easy, tho' the work itself vastly more tedious than other Methods which have been discover'd since, by means of a deeper Application of the *Algebraick Art* than my limits allowed me to use here. My Purpose is however sufficiently answer'd; for as every one who would understand the Reason and Use of *Logarithms*, is not under any necessity of constructing them, that being often done already; so I design'd chiefly what I think is most generally demanded, that is, (1.) To demonstrate the *Origin and Nature* of those Numbers, or shew that there are really such Numbers to be found, as we define *Logarithms*; which could not be better or more naturally done than by the Method of the Inventor. And then, (2.) to explain and demonstrate their Use and Application; which is the same, whatever way they are calculated or constructed.

I shall say but this one thing more, *viz.* That as those other Methods of Construction are chiefly owing to Sir *Isaac Newton's* Binomial Theorem; so far as I have explained that Theorem, (which is only so far as relates to *simple* or *proper Powers*, i. e. having Integral Indexes) I have so far also made their way easy, who would study those other Rules of Construction wherein that Theorem is also apply'd to *Roots*; which Rules they will find no where more easily and fully explain'd than in *Ronayne's* Algebra.

6. Of the *Combinations* of Numbers, a Part of Arithmetick which has been but very little and generally handled by our *English* Writers; and as little by others that have fallen in my way. We have indeed most of the fundamental Propositions of it, in *Hill's* Arithmetick, yet far short of the Length I have carried it to here. As the thing is in itself curious, and not without considerable Use, especially in the Calculations of Chances, I have explain'd it the more particularly. Here also you have another Demonstration of the *Binomial* Theorem for *Coefficients*.

B O O K VI.

Contains the Application of the Doctrine of *Proportion* to the Common Subjects of Human Affairs: Wherein I have gone thro' a large and complete Course of all the Common Rules and Branches of this Application. I have labour'd to make the Rules as plain and intelligible as possible; and at the same time express them so, as the vast

Ex-

Extent of their Use may easily appear; and young People may not be so limited in their Notions of those things, as not to be able to go further than the few Examples to which they have found them apply'd in Books, or by their Teachers; or such Examples as are strictly of the same kind, and proposed in the same manner with these: as I have often found to be the Case in the Course of my Business and Experience in those Matters. The best Remedy of which, is to understand the Reason and Demonstration of every Rule, and see the Applications of it in a great Variety of Subjects and Circumstances. Therefore I have shewn the Reasons of all the Rules by their Dependence upon the preceding Theory. But lest any thro' neglect, or some other fault, should not understand that Theory, I have here given some other Demonstration of the chief and most useful of those Rules. And to make the Application complete, I have given you not only a sufficient Number of Questions of common Use and Occurrence in every Branch; but also a great many that are uncommon and curious, the studying of which will serve this very useful Purpose, *viz.* to lead one to a clearer and readier Apprehension of the Application of the Rules of Arithmetick, and especially of Proportion, which is the most important and difficult thing in the Practical Arithmetick.

The Applications relating to the *Interest* of Money being of great Concernment to Society, I have explained and demonstrated those at large. And here I found myself necessarily engaged in the Examination of a Question wherein Sir *Samuel Moreland* and Mr. *John Kersey* have widely differ'd. The Question is about the Calculation of the present Worth of an *Annuity* to continue any number of Years, discounting simple Interest. Whether Sir *Samuel's* Book, which he calls *The Doctrine of Interest*, wherein he finds fault with Mr. *Kersey's* Rule, which is in his *Appendix to Wingate's Arithmetick*, was written before *Kersey's* Death, or whether he ever saw it, or gave it any Answer, is what I know not; but the Difference seem'd to me too considerable to pass over. Upon the most careful Examination, I was determin'd to Mr. *Kersey's* side: tho' I was very soon afterwards surpriz'd to find my Opinion contradicted by the Mathematical Writers, who have taken the other side, and form'd their Rules upon *Moreland's* Foundation; as particularly by *Ward*, with this Remark, that "*Moreland* has detected several material Errors committed by *Kersey* upon *Wingate*." This put me upon a more narrow Examination of the Question, which ended in a further Confirmation of my former Opinion; and yet what Mr. *Ward* and others have done merely as Mathematicians, is right; they have assum'd a certain State of the Question, and upon that rais'd Rules which are good upon the Justice of that State of the Question, but not otherwise: and therefore in the Defence of Mr. *Kersey's* Rule, I differ from *Ward*, and others who have taken that Method, not as to any pure Mathematical Truth in Numbers, but merely as to a point of Right and Equity, in the manner of stating a Question betwixt Man and Man, according to the Conditions previously agreed to: But that every body may judge and chuse for themselves, I have given the Rules and Reasonings upon both sides.

As to the Rule of *Position*, or *Falshood*; which is common enough in Books of Arithmetick, I have omitted it, because it is of little or no Use in real Business, and very limited in its Application: Besides, whoever has the least Smattering of the Algebraick Method of solving Questions, can do all, and much more, than this Rule teaches.

Of the Method of Demonstration used in the following Work.

I have every where endeavour'd to take the most easy and natural way the thing would admit of. In the first and simple Parts there is but one way almost to be taken; but in the more complex and difficult Parts, as there is room for a Choice, I have generally used the *Algebraick* Method, as what is natural and proper to *Arithmetick*, and the most easy and plain Method that can be used in most Parts of this Science, and without which many useful and curious things could not be demonstrated. I have not supposed the Student of Arithmetick already acquainted with Algebra; but have gradually explain'd the

the Principles and Rules of it, as far as my Purpose requir'd. As *Algebra* is nothing else but an universal Method of representing Numbers, and reasoning about them, so it very naturally belongs to Arithmetick: And in the Opinion of the Great Sir *Isaac Newton*, who calls it the *Universal Arithmetick*, makes, with what in distinction from it he calls the *Vulgar Arithmetick*, but one complete Art of Computation. But my Design not reaching to a complete System of Arithmetick in this larger Sense, I have done no more as to *Algebra* than is necessary for demonstrating the System of Arithmetick in the more strict Sense.

I have indeed been ask'd, why any thing is brought into a Treatise of Arithmetick, which stands in need of the *Algebraick* Art, or can be better done by that means than otherwise, and not rather refer'd to a Treatise of *Algebra*? The Answer was obvious, *viz.* That wherever these things are placed, they belong to the System of Arithmetick: And for the *Algebra* requir'd to the Demonstration, if one has already learnt it in a more express and particular Study of that Art, it is well; but if not, 'tis just as proper and easy to learn it in a Course of Arithmetick, as naturally belonging to this Science. And if it is again ask'd, Why then I have not extended this Work to all the Parts of the *Algebraick* Art, and thereby made a System of Arithmetick more complete? I answer, That having the Choice of my Subject, I have given it such limits as I thought convenient, and done such a Work as I thought was most wanted: Those who incline to make a more particular Study of the *Algebraick* Art, must seek it elsewhere. But if what is done here, both as to the Principles and Application of *Algebra*, be well understood, it will, I believe, prove an useful Introduction to the higher Parts of this admirable Art, and a powerful Incitement to the further Study of it; when it is consider'd, how the most simple Elements of it are sufficient for acquiring such a Knowledge of Arithmetick as can't be obtain'd without it in many things, and in others not without much greater difficulty.

I hope then there will no Discouragement arise from a Prospect of Difficulty in this Method, by such as are willing to study Arithmetick in a reasonable manner: For tho' there are difficult and abstruse things in the *Algebraick* Art, yet all the Principles and Rules of it used in this Work, are in effect no more than a particular kind of Language; or rather a compendious way of representing and comparing Numbers and the Effects of their Operations: which may be learnt with a little pains, in two Lessons, or three at most; and as they are explain'd and apply'd by degrees, it will become easy and familiar as you proceed.

For those who would study only the Practical Part, without the Theory and Reasons of Things; they will find what they want in the first, second, and sixth Books, with the second Chapter of Book III. In all which, let them pass over the *Demonstrations*. And if they would go further, they may read the *Problems* in the fourth and fifth Books.

It remains that I explain the Meaning of a few Names used in the following Work, for different kinds of Propositions.

A DEFINITION is the Explication of the Use and Meaning of any Word or Term of Art; as of this itself and the following.

AN AXIOM is a Proposition whose Truth is self-evident.

A THEOREM is a Proposition whose Truth is to be demonstrated.

A LEMMA is a Proposition to be demonstrated; and which is premised to some other, to serve as a Principle for the more easy Demonstration of this other.

A PROBLEM is a Proposition of something to be done or discover'd.

A COROLLARY is a Proposition gain'd in consequence of another, whose Truth is evident from the Truth or Demonstration of that other.

A SCHOLIUM is some further Explication relating to what precedes.

Observe, In the *Demonstrations* of the following Work, when any former Proposition is quoted, it's understood to be in the same Book and Chapter in which it is quoted, unless it is otherwise expressed.

A

SHORT HISTORY

O F

ARITHMETICK.

THAT Arithmetick was very early in the World, no body can doubt, because the Idea of Number arises from all things about us. In the beginning, while the Way of Living was simple, and things were in a manner common, the Knowledge of Numbers would make a small Progress: But when *Property* and *Commerce* began to be established, Men would soon find the Necessity of enquiring into the Nature of Numbers, and contriving an *Art* of *Numbering*; without which no Business can be carried on. This was, no doubt, very rude at the first, and improved by degrees; as all our Knowledge is: But where, and by whom, Arithmetick received its first Form of an *Art* or *Science*, we know little about it. If the *Phœnicians* were, as it is conjectured, the first Merchants after the Flood, (and before that we know nothing of the Affairs of Mankind) then it is probable, the *Art* began among them; by whom *Trade* and *Arithmetick* were carried into *Egypt*; and here, 'tis thought, began the *mystical* Application of Numbers: For the *Egyptians* explained every thing by these; the Nature of the Gods, of *Human Souls*, the *Virtues*; in short, for every thing *divine* and *human*, they found some Symbol or Representation in Numbers: Hence we hear of the wonderful Virtues and Properties of some particular Numbers, as *One*, *Two*, *Three*, *Four*, *Seven*, and *Nine*. From *Egypt* this Knowledge passed into *Greece*, which added its own Improvements to the mysterious Part; of which a great deal is to be seen in *Plato*; the Life of *Pythagoras* by *Jamblichus*; and more lately in the Commentators upon *Boethius's* Arithmetick. Now we are come to the Country where we may expect to find the first distinct Rudiments of the Science.

The first thing Men were obliged to do to make their *Ideas* and Knowledge of Numbers useful in Society, was to establish some Method of *Notation*, and then upon this found an *Art* of *Computation*: after this they would gradually enquire into the *Relations* and *Properties* of Numbers; and so the Science went on.

The *Greeks*, *Hebrews*, and other Eastern Nations, used a Notation by the Letters of their Alphabet. The *Greeks*, particularly, had two different Methods; the first was much the same with the *Roman* Notation, explain'd in *Chap. 2. Book I.* of the following Work, which is a very imperfect Method. Afterwards they had a better Method, in which the first nine Letters of their Alphabet represented the first Numbers from One to Nine, and the next nine Letters represented any Number of Tens from One to Nine, that is, 10, 20, 30, &c. to 90. Any Number of Hundreds they expressed by other Letters, supplying

supplying what they wanted with some other Marks: And in this Order they went on, using the same Letters again with some different Marks to express *Thousands*, *Tens of Thousands*, *Hundreds of Thousands*, &c. As to this Method, 'tis to be observ'd, that they were upon the very Point of discovering the *Arabian* Notation: For, as they made the Progression to 9, they wanted but one Step further, *viz.* Instead of using other 9 Letters, to make the same 9 change their Values in a decuple Progression according to their Places, which would in course discover the Necessity of a Character that of itself signifies Nothing, only fills up a Place.

The Manner of their *Computation*, (*i. e.* of *Addition*, *Subtraction*, &c.) and the Difficulty of it, especially in great Numbers, we may easily discover from the *Notation*. As to any express Treatises upon the *Art of Computation*, they have left us none. There is a Commentary by *Eutocius*, upon *Archimedes's* Treatise of the Dimensions of a Circle; and some Fragments of *Pappus*, in *Dr. Wallis's* Works, which relate particularly to the Work of *Multiplication*, and shew us the great Difficulty of their Practice, owing to the imperfect Notation.

The most perfect Method of *Notation*, which we now use, was owing to the Genius of the *Eastern* Nations; the *Indians* being reckoned the Inventors of our Notation; which we call the *Arabian*, because we had it from them, and they from the *Indians*, as themselves acknowledge. When the *Indians* invented this Method, and how long it was before the *Arabs* got it, we cannot tell: These things only we know, 1. That we have no ground to believe, the antient *Greeks* or *Romans* knew any thing about it: For *Maximus Planudes*, the first *Greek* Writer who treats of *Arithmetick* according to this Notation, lived about the Year of Christ 1370, as *Vossius* says; or about 1270, according to *Kircher*; long after the *Arabian* Notation was known in *Europe*: And owns it for his Opinion, that the *Indians* were the Inventors, from whom the *Arabs* got it, and the *Europeans* from them. 2. That the *Moors* brought it into *Spain*; whither many learned Men from other Parts of *Europe* went to seek that, and the rest of the *Arabick* Learning (and even the *Greek* Learning, from *Arabick* Versions; before they got the Originals themselves) imported there by the *Saracens*. As to the Time when this new Art of Computation was first known in *Europe*, *Vossius* thinks it was not before the Year 1250; but *Dr. Wallis* has, by many good Authorities, proved that it was before the Year 1000; particularly that a Monk called *Gerbertus*, afterwards Pope by the Name of *Sylvestre II.* who died in the Year 1003, was acquainted with this Art, and brought it from *Spain* into *France*, long before his Death. The Doctor shews also, that it was known in *Britain* before the Year 1150, and brought a considerable length, even in common Use, before 1250, as appears by the Treatise of *Arithmetick* of *Joannes de Sacro Bosco*, who died about 1256.

Tho' the numeral Figures which we now have are a little different from what the *Arabians* use, having been changed since they came first among us; yet the *Art of Computation* by them is still the same.

Having said all that's necessary about the *Notation* of Numbers, we shall go back again, and see what kind of *Science of Arithmetick* is to be found among the Antients, with the Progress of it till now.

The oldest Treatise extant upon the *Theory of Arithmetick*, is *Euclid's* 7th, 8th, and 9th Books of *Elements*; wherein he gives us the Doctrine of *Proportion*, and that of *Prime* and *Composite* Numbers. Both of which have received Improvements since his time, especially the former. The next, of whom we know any thing, is *Nicomachus* the *Pythagorean*, who wrote a Treatise of the *Theory of Arithmetick*, which consisted chiefly of the Distinctions and Divisions of Numbers into certain Kinds and Classes, as *Plain* and *Solid*, *Triangular*, *Quadrangular*, and the rest of the Species of *Figurate* Numbers (as they called them) Numbers *Odd* and *Even*, &c. with some of the more general Properties of the

the several kinds. As to the time in which *Nicomachus* lived, some place him before *Euclid*; others long after. His *Arithmetick* was published at *Paris* 1538. What kind of Work it is, we may guess by the *Latin* Treatise of *Arithmetick* of *Boethius* the Philosopher, who lived at *Rome* in the time of *Theodorick* the *Goth*; and is the next remarkable Writer extant upon this Subject. He is supposed to have seen and copied most of his Work from *Nicomachus*.

From this Work of *Boethius*, with a few small Abstracts of the same nature, made very long after his Time, as that of *Pfellus*, and *Jodochus Willichius*, both in *Greek*; some have said that the antient *Arithmetick* consisted of nothing else but these Divisions and Distinctions of Numbers. I confess I was surprized to find this Account from such an Author as *Wolffius*, to whom *Euclid* is no Stranger; whose Books contain things much more important in the Science of *Arithmetick* than these Distinctions; and want many of them, that are in *Boethius*: For *Euclid* speaks nothing of the *Figurate Numbers*, and their various Species and Classes; except what relates to Squares and Cubes. And, on the other hand, *Boethius* has very little of *Euclid's* Doctrine.

We must come next to the Times when the *Arabian* Notation was known in *Europe*; after which we find many Writers both upon the *Theory* and *Practice*. The oldest of them, who is very considerable, is *Jordanus* of *Namur*, who flourish'd about 1200. His *Arithmetick* (from which I have taken several things) was published and demonstrated by *Joannes Faber Stapulensis* in the fifteenth Century, (who has given us himself a Compendium of *Boethius*) soon after the Invention of Printing. It's altogether upon the *Theory*; and contains most of what *Euclid* and *Boethius* have, and many other curious Theorems. The same Author wrote also upon the new Art of Computation by the *Arabick* Figures, and called this Book *Algorismus Demonstratus*; the Manuscript of which, Dr. *Wallis* says, is in the *Savilian* Library at *Oxford*. But it has never been printed, as I know.

As Learning advanced in *Europe*, so did the Knowledge of Numbers; which by degrees received large Improvements both in the *Theory* and *Practice*, owing in a great measure to a more perfect Notation. To trace out every Step in that Improvement, is impossible; therefore I shall only name a few of the remarkable Writers after *Jordanus* and *Sacro-Bosco*, both named already. As to the Writers, these were most remarkable in *Italy*, viz. *Lucas de Burgo*, about the Year 1499, whose *Arithmetick*, which is both Theoretical and Practical, Dr. *Wallis* commends much: *Nicholas Tartaglia*, whose Work is chiefly Practical. He is called by some the Prince of the Practitioners; which must be understood only for his own Times. In *France*, there were *Clavius* and *Ramus*. In *Germany*, *Stifelius* and *Henischius*. In *England*, *Buckley*, *Diggs*, and *Record*. All these, and many more, were before the Year 1600. But since that, our Writers are almost innumerable.

As to the Improvements made since the *Arabick* Notation was known in *Europe*; besides many things in the *Theory*, particularly in the Nature of *Progression*, both *Arithmetical* and *Geometrical*, in the Nature of *Powers*, and in the *Extraction of Roots* and the *Combinations of Numbers*, which we do not so well know the History of; there are a few very considerable Improvements, in the practical Part, of which we can give a better Account. But that I may connect the Antient and Modern History, we must go back to the second Century of *Christianity*, in which *Claudius Ptolomeus* lived, who is supposed to be the Inventor of the *Sexagesimal Arithmetick*; which was a new Method of Notation, and consequently of Computation, designed to remedy the Difficulty of the common Method, especially with regard to *Fractions*. The Nature of it was this: Every Unit was supposed to be divided into 60 Parts, and each of these Parts into 60 Parts, and so on; hence any Number of such Parts were called *Sexagesimal* Fractions. And to make the Computation in Integers also more easy, he made the Progression in these also *Sexagesimal*. Thus, From one to fifty-nine were marked in the common way; then sixty was called a *Sexagena prima*, (or first *Sexagesimal* Integer) and marked with the Sign of Unity

and one single Dash over; so sixty was thus expressed I' . Two sixties, or 120, thus II' ; and so on to 59 times 60, (or 3540) which is LIX' . Then for 60 times 60, (or 3600) this he called a *Sexagena secunda*, (or second *Sexagesimal* Integer,) and marked any Number of them less than 60, by the Signs of Numbers less than 60, with two Dashes: Thus, 60 times 60 (or 3600) was marked I'' ; two times 3600, thus II'' ; ten times 3600, thus X'' ; and so on to 59 times 3600. In this manner the Notation went on: And when a Number less than 60 was joined with any of these *Sexagesimal* Integers, their proper Expression was annexed without the Dash: Thus, the Sum of 4 times 60 and 25 is expressed thus, IV', XXV . The Sum of twice 60, ten times 3600, and 15 is expressed X'', II', XV ; the highest *Sexagesimal* being set next the Left-hand. As for the *Sexagesimal* Fractions, they were marked the same way, their Numerators by the Signs of Numbers less than 60, and their Denominators by one or more Dashes (according as they were Primes, Seconds, &c. i. e. 60, 3600, and so on in the order of the Powers of 60) set either over the Numerator upon the Left-hand, or under it upon the right. Thus five sixty Parts are marked V' or V_60 . And fourteen 3600 Parts XIV'' or XIV_{3600} . The Practice by this Notation would be easier than their common Method; yet still very difficult, especially in Multiplication and Division, as appears by the Work of *Barlaamus Monachus*, called *Logistica*; wrote in *Greek* about 1350; translated into *Latin*, and published 1600. And here it is remarkable how very near this Method is in the general Nature of it to the *Arabick*. He wanted no more, but instead of *Sexagesimal* Progression, to make it *Decimal*; to make the Signs of Numbers from one to nine simple Characters; and lastly, to make a Character which signifies nothing by itself, serving only to fill up Places. But every Age and Nation has its Genius; and therefore we owe this to others.

As this *Sexagesimal* Notation was used chiefly in the Astronomical Tables, so for the sake of these, it was not laid aside immediately after the Introduction of the *Arabick* Notation. The *Sexagena Integrorum* went first out; but the *Sexagesimal* Fractions continued till the Invention of the *Decimals*. *Regiomontanus* about the Year 1464, is the first we know who in his *Triangular Tables* divided the Radius into 10,000 Parts instead of 60,000; and so tacitly introduced decimal Parts in place of *Sexagesimals*. *Ramus* in his *Arithmetick*, written about 1550, (and published by *Lazarus Schonerus* in 1586) uses decimal Periods in carrying on the Extraction of Square and Cube Roots to Fractions. The same did our Country-men *Bucklaus*, before *Ramus*; and *Record* about the same time. But the first who wrote an express Treatise of Decimals, was *Simon Stevinus*, about 1582.

As to the *Circulating Decimals*, *Dr. Wallis* was the first among us who took much notice of them. But I have spoke of this already.

Another most wonderful Improvement that the Art of Computation has received from the Moderns, is the *Logarithms*; the unquestionable Invention of the Lord *Neper*, Baron of *Merchiston* in *Scotland*, towards the end of the sixteenth Century, or beginning of the seventeenth.

Dr. Wallis is the Author of the *Arithmetick of Infinites*; which has been very usefully applied in *Geometry*.

But the Consummation of the Art is in the *Algebraick* Method of resolving Questions: The particular History of which, I have said nothing of; because, tho' *Algebra* belongs to *Arithmetick* in a larger view, yet I have limited myself to *Arithmetick* taken in a more strict sense, as it is distinguished from *Algebra*: Therefore I shall only say, that most of the Authors mentioned have also wrote upon the *Algebraick* Art, which came into *Europe* at the same time, and by the same hands, as the *Numeral Notation*: *Lucas de Burgo* being reckoned the first *European* Writer on this Subject.

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ARITHMETICK

BOOK I. Of WHOLE NUMBERS.

CHAP. I.

Of the Nature of ARITHMETICK in general, as to its Object and Operations; with the Division and Order of the Science.

§. I. DEFINITIONS.

I. **A**RITHMETICK is the Science, or Knowledge of Number; which is either *Unity*, or a *Multitude* of Units.

II. Of *UNITY*. When we consider any thing by itself alone, either as indivisible, or at least undivided; or also considering several things as connected in some certain manner, thereby making up a whole, neglecting what differences may be among them in other respects; the Idea we have of this thing, or Collection of things, consider'd in this manner, is called *Unity*, or *One*, *i.e.* an individual thing of a particular Kind and Name; as one Man, one Stone, one Kingdom, one Army.

III. Of *MULTITUDE*. When we consider several things as really distinct Individuals, and which separately taken we would call Units, whether they are of the same, or of different kinds and natures of things; or whether they are really separated from one another, or only distinguished by the Imagination, as the conceivable Parts of any continuous Body (for Example, a Rod) looking at no more in them but that they are not the same individual thing; the Idea acquir'd by this way of considering them, is called *Multitude*. (or *many*, in distinction from one;) so we say, a Multitude of Men, of Horses, of Trees.

SCHOLIUM I. The Units, or Individuals that make a Multitude, may either be of the same Kind, or Species and Denomination of Things, or they may not be so: for however different several things are in Name and Nature, the Idea of Multitude arising from them

is the same: So, for Example, a Man, a Tree, and a Horse, make as truly a Multitude, as if they were all Trees, or Men, or Horses: They are at least a Multitude of Beings, or Things; for the Idea of Multitude has no dependence upon the Likeness of the Things from which it is formed, but only upon their different and distinct Being or Existence, in whatever manner they are connected together, or under whatever other differences they really exist, or are conceived to exist.

SCHOLIUM 2. We may also conceive Multitudes of things under the Notion of one, or many; and so we may say one Multitude, or a Multitude of Multitudes. But it is to be observ'd, that in this case, the Multitudes which make the Parts of one Multitude, are conceiv'd each as an Unit, or one of its own Species, (*viz.* Multitude,) to distinguish it from the Multitude of which it is one constituent Part: so that Multitude in its general Nature is still a Collection of Units, which are in all cases simple Units in respect of the Multitude which they compose; tho' they may be themselves Multitudes composed of more simple Units. And this Distinction of Units may be very well distinguish'd by the Names, *Simple and Collective Units*.

IV. Of NUMBER. *Unity and Multitude* comprehend the whole Object of Arithmetick, and are both comprehended under one general Name, *Number*; whose Definition does therefore take in the other two; and may be made thus, *viz.* *Number* is the Name of that Idea or Notion under which things being consider'd, they are said to be *One* or *Many*: Every particular *Multitude* having a distinct Name; as two, three, four, &c. As afterwards will be taught.

SCHOLIUM. We must make a little Stand here, and take notice of an old Dispute among *Arithmeticians* about the Definition of Number; some denying *Unity* to be a Number, and others affirming it: about which there has been a great deal of Argument fill'd with abundance of idle and nonsensical Jargon, to the shame even of some late Writers: For after all the learned Contention, it dwindles into a meer Dispute about the Name, or what shall be the Use of the word *Number*; which no doubt each Party has a Right to establish for themselves at pleasure; but no Right to impose it upon others: And so then where is the ground of a Dispute? For if any Man asks me whether *Unity* is a Number, I must first know of him what he calls a Number, and then I answer him according to his own Definition; or I first give him my definition of the word *Number*, and then answer his Question out of that. But we shall hear their different Definitions: Some define *Number* a *Multitude* of Units; and according to them it is plain, *Unity* is not a Number in that sense in which *Unity* and *Multitude* are distinguished, (for we have observed already how *Unity* and *Multitude* may be applied to the same Subject in different senses;) so that these by denying *Unity* to be a Number, do only deny it to be a *Multitude* in the same sense or application in which it is *Unity*, which no body will affirm. Others adhere to the former Definition which comprehends *Unity* and *Multitude*; but some of them are as much in the wrong, because they contend about it as if they had the only right to settle the Use of Words; and still they are more ridiculous to pretend they are arguing about the Nature of Things themselves, when it's only about a Word: and if it does not yet appear that there can be no more in the Dispute, let this be consider'd, *viz.* That *Unity* and *Multitude* are agreed upon to signify different things. And I believe it must be yielded, that these comprehend the whole Object of Arithmetick; therefore *Number* must either signify the same with one of these, or be apply'd as a general Name to both; and then the only remaining Question will be, Which is most reasonable? And this, I think, will be easily decided by considering, that of two Words merely synonymous, one is superfluous; but it's often very convenient to comprehend several things, which have also their different Names, under one general Name, because of some common thing in which they agree, as it is in this present case. And those who would make *Number* equivocal with a *Multitude*, are press'd also with this other Difficulty, *viz.* That if they retain their

Defi-

Definition of *Arithmetick*, viz. the Science of Numbers, then *Unity* will be no part of the Object of *Arithmetick*, since it is not a Number. But this, I believe, they will not say; for whatever can be any part of the *Data*, or Means by which a Question in *Arithmetick* is solved, or be itself a real and positive Answer, must belong to the Science, as a part of its Object. And indeed, tho' *Euclid* defines Number to be a *Multitude of Units*, yet all along he treats of *Unity* under the same Name.

V. Of NUMBERS, *Abstract and Applicate*. 1. When in things number'd we consider their Number, abstracting from (*i. e.* not attending to) their other particular Properties and Differences; the Idea or Notion we hereby form of Numbers, is called *abstract* or *general*; or, we are said to consider Number *abstractly*: Because whatever is true of that Number of things consider'd simply and purely in the Number, is true of the same Number wherever it is found, or in whatever things it exists.

SCHOLIUM. We can form no Idea of Numbers, without that of things number'd; because it is an Idea form'd by comparison of things: Yet while we consider a Number of particular things, tho' we still know that the Number is inseparable from other Ideas that make up the complex Idea of these things, it's in our power to consider and compare only the Numbers of things together, and examine their various Properties and Differences; and the Mind can perceive at the same time, that whatever is true of the Number of these things, must necessarily be true of the same or equal Number of whatever other things, wherein the Number only is what we consider and compare. From whence it is that we speak of Numbers without naming any particular things; by barely naming the particular Number, or joining it with the general word Thing, (which is always supposed, when not mentioned.) So we speak of the Number, Two, Three, &c. *i. e.* two, three things, without pointing out any particular thing: Because where nothing is taken into consideration (as the Subject of Comparison and Reasoning) but the Number of things, then any things may be supposed; and so tho' the Names *Two* or *Three* are Names of particular Numbers, inseparable from particular things, yet because the same Numbers in every other thing must have the same Names and Properties, we make use of the Name without mentioning particular things: not because that Name belongs to (or represents) an Idea of that particular Number which is not connected with any particular things; but because it is a general Name applicable to the same Number of whatever particular things; and is used in this manner without mentioning any thing, when it's indifferent which things are supposed, (*i. e.* when the Number only is the matter in question;) in the same manner as we have here used the word *Number* itself, without mentioning a particular Number, as *One*, or *Two*, &c. Not as if the word *Number* represented an Idea different from all Particulars; but as it is a general Name comprehending them all.

2. When we consider Number not in its general Nature, as above explained, but as it is a Number of certain particular things, as two Years, two Men, or two Yards; then we call it an *Applicate* Number: which Name I chuse, for its obvious Meaning, rather than the word *Contract* or *Concrete*, which some Authors use.

SCHOLIUM. When particular things are mentioned, there is always something more considered, than barely their Numbers; so that what is true when Numbers are compared in their abstract or general Nature, (*i. e.* when nothing but the Number of things is consider'd) will not be true, when the Question is limited to particular things: So, for example, the Number *Two* is less than *Three*; yet two Yards is a greater quantity than three Inches: for the Comparison here is not simply of the Number of things two, and three; but of the Numbers joined with another Consideration, viz. that of their lengths. And when things are of quite different Species, then tho' we can compare their Numbers *abstractly*, yet we cannot compare them in any *applicate* Sense. And this Difference is necessary to be consider'd, because upon it the true Sense, and the Possibility or Impossibility of some Questions depends; as we shall learn more particularly afterwards.

COROLLARIES.

1. Number is unlimited in respect of Increase; so that beginning at Unity, and adding to it another Unit, and to this last Collection another Unit, and so on, we may proceed *in infinitum*, *i. e.* we can never come to an end, or never conceive a Number, but still there is a greater. But on the side of Decrease it's limited; Unity being the first and least Number, below which therefore it cannot descend. In what sense Unity is said to be divided into Parts, which make a greater Number, shall be consider'd in its place. Also we may not only begin at Unity, but at any other Number, and increase it *in infinitum*, by the continual joining of Unit after Unit; or diminish it to nothing, by continually retracting or taking away Unit after Unit.

2. Any Number may be increased by any other Number, or by any Number of Numbers; for every Number is either Unity, or a Collection of Units, which can be joined separately to another Number till they be all joined. Also it may be decreased by any Number not greater than itself; or by any two or more Numbers, which taken all together do not exceed it, *i. e.* such a Number or Numbers may be taken out of it.

3. Every greater Number may be consider'd as compos'd not only of Units, (which are its most simple constituent Parts) but also variously of other Numbers lesser than itself, according to the variety of lesser Numbers, whose Units taken all together make a Collection equal to that Number; or, according to the various Distributions that may be made of its Units, by putting them together in separate Collections: where also every lesser Number may be conceived as a Part of every greater; which is as a Whole with respect to all the lesser Numbers, which, being joined together, make up that Number.

§. 2. *Of the general Division, and Order of this Science.*

THE most general Division of Arithmetick is that of *Theory* and *Practice*.

The *Theory*, or Speculative Part, is that Science which considers and explains the Properties and Relations of *Pure and Abstract Numbers*; consisting of such Propositions as exhibit to the Understanding certain Truths concerning Numbers, either more general or more particular; as *Axioms* and *Theorems*.

The *Practical* Part is the Art of Numbering, or applying the Theory to the Solution of Questions, either in abstract or applicate Numbers; consisting of *Problems*, or such Propositions as require something to be done or effected: and gives us a Rule for the Performance; teaching how, by means of certain known Numbers, to discover other Numbers connected and related to them, according to the Conditions proposed in the Question; or at least to find that from the given Numbers, compared and applied to one another, as the nature of the Question requires, there arises no Number.

Observe, Some consider as Theory all that is propos'd in abstract Numbers, whether Theorems or Problems; and the Application to Questions in applicate Numbers only, they call the Practical Part. Others define the Theory as above; and confine the Practical Part to Problems of Abstract Numbers: and Problems of Applicate Numbers they call the *Effective* Part.

As the Truth and Reasons of the Practical Rules are contained in *Theorems*, with other more general Principles, as *Definitions* and *Axioms*, so they are to be reckon'd Deductions from them, or rather their Applications: And therefore in the natural Order, the Theory ought to precede the Practical Part. But yet these two Parts ought not, and cannot be treated entirely separate from one another; *i. e.* all of the first Kind together, and afterwards all of the other: But they must be mix'd together according to their Dependence. It's certain that *Theory* must precede *Practice*, because that contains the Grounds and Reasons of this; yet 'tis as true, that we can make but a small Progress in Theory, till we under-

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understand the fundamental Elements and Rules of the Practical Part: These being indispensibly necessary both for understanding the Sense, and illustrating, or proving by Examples, the Truth of Theorems.

Wherefore the *Division* that must be followed in explaining this Science, is not that of *Theory* and *Practice*, (tho' these must also be disposed according to Reason, and their natural Connection and Dependence;) but the most proper and reasonable *Division* is into two other Parts, under the Titles of the *Simple* and *Comparative Elements*.

If we reflect upon the Definitions and Corollaries already explained; these are so many of the first general Principles and Axioms of the Science: And from these we shall easily understand the Reason of this Division, and what in general belongs to each Part. For it's plain, that the most general, and what we may call the only *absolute* Property of Number, is, a *Capacity of Increase* in infinitum, or *Decrease to nothing*: all other particular Properties are *relative*, depending upon the Comparison of *Numbers* together. And since there is nothing in *Numbers*, but different Collections of Units, or different Compositions of lesser *Numbers* in greater, these particular Properties must all depend upon the Effect of different Applications of *Numbers* to one another, whereby they are variously compounded together or resolved, according to certain Conditions. And for Arithmetical Problems, or Questions, in which an unknown Number is to be found by means of certain Connections and Relations it has to some known Numbers; these Connections can consist in nothing else but this, *viz.* That the Number sought is the Result of variously increasing and decreasing the known Numbers by one another, according to the Conditions proposed; so that all that can be known or done in Arithmetick does evidently relate to, and depend upon, the Application of Numbers to one another by Composition and Resolution, or Increasing and Decreasing them.

Therefore the first and fundamental Part of Arithmetick is the Knowledge of the various Rules and Operations (with the Principles upon which their several Reasons depend) by which Numbers are compounded and resolved; *i. e.* increased and decreased by one another, which are the fundamental Elements of Practice; including in general all that can be done with Numbers; and indispensibly necessary also for understanding and proving the more particular Theory; which does all relate to the Effect of these Operations: Which I have therefore justly, I think, consider'd as the *Simple* and Primitive Elements of Arithmetick. What further Subdivision of this is necessary, shall be shewn in its proper place.

All the rest of the *Science of Arithmetick* I comprehend under the general Name of *Comparative Elements*; because it consists of such relative Properties as arise from the comparing of Numbers together, and applying them to one another by the various Methods of Compounding and Resolving, taught in the first Part; as also the Solution of such Questions as depend upon these Relative Properties.

In the remaining Part of this Book, with the second and third, you have the first Branch, or Simple Elements explained; and the Comparative Elements in the remaining Books.

§. 3. Of the Operations of ARITHMETICK in general.

BY what has been already explained, it will be obvious, That all the *Operations* in Numbers are of two Kinds in General, *viz.* *Augmenting* and *Diminishing*. Each of these are performed after two different ways, and thereby come under two different Names: Thus, *Augmenting* is divided into ADDITION, and MULTIPLICATION; *Diminishing* into SUBTRACTION, and DIVISION: Which shall be explain'd in order. Some add a third Class, whose Branches are call'd *Involution*, and *Evolution*; or also *Raising of Powers*, and *Extracting of Roots*. But these may be comprehended under *Multiplication* and *Division*; for the Operation is of the same general Kind, only under certain Limitations. They will deserve however to be explain'd distinctly by themselves.

But

But there is yet something previous to all this, *viz.* The Knowledge of the Signs whereby our Ideas of Numbers are expressed or represented; for without some easy Method and Art of representing them, so as they may be clearly and distinctly compared, there could be little or nothing known or done in Numbers. This Art we call the *Notation* of Numbers; which, if we consider by itself, and in its primary Design, is only a necessary Instrument for the better and more easy Comparison of Numbers, and performing their Operations; and is therefore rather a Handmaid, than an essential Part: For we can call nothing essential but what belongs to the very Nature of a Thing; and which being taken away, the thing would be destroy'd: not that which is arbitrary, and may be changed, and another thing put in its place at pleasure, as it is in the Notation of Numbers. However, as there must first be a Method of Expression instituted, whatever that is, so far as the Rules of Operation depend upon it, it is the Foundation of them; and therefore it is commonly look'd upon as the first Rule or Operation in Arithmetick, making in all five fundamental Operations, *viz.* *Notation*, *Addition*, *Subtraction*, *Multiplication*, and *Division*. But still it ought to be consider'd only as an arbitrary Rule and Foundation, which requires and supposes no other Principle but this, *viz.* That *any of our Ideas may be represented by any Marks or Signs we please to institute*: whereas the other Operations, besides what they owe to the Notation, have also a dependence upon Reasonings from the Nature of Numbers themselves.

Before we enter upon these Operations, we must here repeat an Observation which has been already made, *viz.* That all Science must begin with *Theory* as a Foundation for *Practice*. Now this Order we have in effect followed; for the Definitions and Corollaries explained in §. 1. are the first and more general Principles of this Science; to which if we join this general *Axiom*, *viz.* That *the Whole is equal to all its Parts*, we have all that is necessary for entering upon these practical Elements. What other Principles are employ'd in particular Rules, shall be explain'd in order as we go on; for they are gained most part by consequence in the progress of the Science.

C H A P. II.

Of the Notation, or Expression of NUMBERS, with their Distinction into Integral and Fractional.

§. 1. *Of the Notation of Numbers.*

D E F I N I T I O N.

NOTATION is the Method or Art of Expressing Numbers: which is done two ways; by certain *Words* or *Names*, and also by certain *Signs* or *Characters*, called *Figures*; the one corresponding to the other in the Representation of the same Numbers, and both equally necessary: The Figures being contriv'd for the easy management of Operations, whereby the greatest Numbers are compared, and the Operations performed with the greatest ease and readiness; without which, our Knowledge in Numbers had reached a very short way. And so much does the Science owe to these, that upon this Account some call it, *The Art of Figuring*; but we might better call it, *The Science of*

of Numbers, as they are represented and managed by Figures: So that this is the principal Branch of *Notation*; which yet cannot be without the other, the Names of Numbers being necessary for our converling or speaking to one another about Numbers, and the Result of the Operations made by the Figures, and for the actual numbering or telling over things, by the Application of Names in an orderly Progression from Unity, still joining one Unit after another to the Collection; or telling things out one by one. For without some Signs whereby the Number, as it increases, is continually distinguished, we could make nothing of the Numbers of things, nor compare one Number with another: and Words are the most proper and convenient Signs for this purpose; which are also of good and necessary Use in making the Operations with the Figures.

The System of the Names of Numbers is indeed a part of our Language, and therefore the Writers on Arithmetick suppose them to be known, and reckon it their business only to explain the Representation of Numbers by Figures, and their Correspondence to the Names: But it will be a more regular and just Method to explain both the Systems of Names and Figures by themselves; and then shew their mutual Correspondence. These two Systems I shall explain under the Titles of the *Nominal* and *Figural Notation*; which being compared, their mutual Correspondence will be easily understood. But this I shall more particularly explain in the Solution of a *Problem*, teaching how from the Expression of any Number in one manner, to find its correspondent Expression in the other.

(I.) NOMINAL NOTATION, or the Expression of Numbers by Words or Names.

A DIFFERENT simple Name for every Number, or even for as many particular Numbers as we have occasion to consider in human Affairs, would be a Burden altogether insupportable; but it is more happily contrived; that a few simple Names, and these compounded together in a very easy manner, answer all the Ends and Purposes both of Speculation and Practice.

The Simple Names of Numbers are these,

		Explanation of this TABLE.
One,	Ten,	One is another Name for <i>Unity</i> : the rest of the Names to <i>Ten</i> express Numbers in a Series from Unity, by the continual joining Unit after Unit; so <i>Two</i> expresses <i>One</i> more <i>One</i> ; <i>Three</i> expresses <i>Two</i> and <i>One</i> , and so on to <i>Ten</i> : a <i>Hundred</i> expresses <i>Ten-Tens</i> ; a <i>Thousand</i> expresses <i>Ten-Hundreds</i> ; a <i>Million</i> is a <i>Thousand-Thousands</i> ; a <i>Billion</i> is a <i>Million of Millions</i> ; a <i>Trillion</i> is a <i>Million of Billions</i> ; a <i>Quadrillion</i> is a <i>Million of Trillions</i> .
Two,	A Hundred,	
Three,	A Thousand,	
Four,	A Million,	
Five,	A Billion,	
Six,	A Trillion,	
Seven,	A Quadrillion,	
Eight,	&c.	
Nine,	...	

So that we have here Names answering to the natural Series of Numbers from *Unity* to *Ten*. But after this, the Series is interrupted, and we pass to the Names of greater Numbers: And all that remains to be explain'd is, how the intermediate Numbers are named from *Ten* to a *Hundred*, and from a *Hundred* to a *Thousand*, and so on. Which is done thus, after *Ten* the Names are compounded of *Ten* and the preceding to a *Hundred*. First, from *Ten* to two *Tens* we proceed thus;

Ten,	Equal to	Ten and One,
Eleven,		Ten and Two,
Twelve,		Ten and Three,
Thirteen,		Ten and Four,
Fourteen,		Ten and Five,
Fifteen,		Ten and Six,
Sixteen,		Ten and Seven,
Seventeen,		Ten and Eight,
Eighteen,		Ten and Nine,
Nineteen,		Two Tens.
Twenty,		

Then from *Twenty* we simply join the Names of the first *nine* Numbers thus, *Twenty-one*, *Twenty-two*, &c. to *Twenty-nine*: then the next Number is *Three Tens*, called *Thirty*; and the same Composition of Names we use from *Thirty* to *four Tens*, called *Forty*: and from this to *Fifty*, (or *five Tens*:) and so on to *Sixty*, (or *six Tens*:) *Seventy*, (or *seven Tens*:) *Eighty*, (or *eight Tens*:) *Ninety*, (or *nine Tens*:) and after *Ninety* and *nine*, *One* added makes *ten Tens*, or a *Hundred*. Then from a *Hundred* (or *one Hundred*) we proceed by joining with it all the preceeding Names from *One* to *Ninety* and *nine*, thus, *One Hundred and one*, *One Hundred and two*, &c. to *One Hundred and Ninety nine*, to which *one* added makes *Two Hundred*. In the same manner we proceed from *Two Hundred*, to *Three Hundred*; and so on to *Nine Hundred Ninety and nine*, and then *one* added makes *Ten Hundred*, or *One Thousand*. In like manner we proceed from *One Thousand*, joining with it all the preceding from *Unity* till we come to *Two thousand*, and so on to a *Thousand-thousands*, or a *Million*; and with this also we join all the preceding Names from *Unity* to *two Millions*, &c. to a *Million of Millions*, or a *Billion*; and so on to a *Trillion* and *Quadrillion*.

Observe, If we would proceed farther, we may use these Names, *Quintillions*, (or a Million of *Quadrillions*;) *Sextillions*, (or a Million of *Quintillions*;) and so on; filling up the intermediate Numbers as before: But for any real Use in human Affairs, we need no Names above Millions. In Mathematical Work, greater Numbers occur; but they are manag'd by Figures, and can be compared without Names; which, if required, may be contriv'd in the manner now mention'd. Observe also, that some instead of the simple Names *Billions*, *Trillions*, *Quadrillions*, &c. chuse the compound Names *Millions of Millions*; *Millions of Millions of Millions*; *Millions of Millions of Millions of Millions*; and so on, compounding the word *Million* once more gradually: But the other simple Names seem more convenient, tho' the contrivance and way of making the complex Names is more obvious and easy; however, since we have little use for Names above Millions, we need not dispute about the difference.

(2.) FIGURAL NOTATION.

AS a few simple Names serve all our Purposes in Arithmetick, so yet fewer simple Figures are found sufficient, not only for common Use, but even to carry us thro' the Infinity of Number: Which Figures, with their corresponding Names, are these;

Figures	0	1	2	3	4	5	6	7	8	9.
Names	Nought	One	Two	Three	Four	Five	Six	Seven	Eight	Nine

That a Figure signifying of itself *Nothing*, (or no Number) is necessary, we shall presently see. In the mean time observe, that the Number of simple Figures being *Ten*, they are hence called the *Digits*, from the Number of *Fingers* (*Digiti*) on our Hands. How this Number came to be chosen, we shall afterwards consider.

All other Numbers greater than *Nine*, (9,) are expressed by Combinations of these *Digits*; placing them together in a Line in various Orders; every Figure changing its Value according to the place it stands in. By this

GENERAL RULE.

In a Rank of Figures placed together in a Line, (reckoning the Order of Places from the Right to the Left-hand; that being the first which is first on the Right-hand; and the second, third, &c. being in order from that on the Left) any Figure in the first place represents the primitive simple Value above expressed, as if it stood alone; and in every place gradually towards the Left-hand, it signifies ten-times as many as it would do in the preceding place.

Thus 1, 2, 3, &c. in the first place signify simply so many Units as above named; but in the second place of a Rank, any one of them signifies ten times so many Units, (or so many times ten.) Ex. 48 signifies forty and eight, [*i. e.* four tens for the value of 4 in the second place, and 8 in the first place.] Again, 60 signifies sixty, [or six tens, which is the value of the 6 in the second place; the 0 in the first place signifying nothing of itself.] Again, in the third place they signify ten times their Value in the second place; or ten times so many Tens, (*i. e.* so many hundreds;) and so on gradually towards the Left-hand, still increasing their value tenfold of that of the preceding place. So 600 is six hundred; 642, is six hundred, forty and two.

Now this general Institution being well conceiv'd, it will, I suppose, be found evidently sufficient for the expression of all Numbers; or that all Numbers in a gradual Series from Unity may be represented thereby. And if this is not evident, it may be made so—Thus; Every Number is either less than Ten; which is expressed by one of the nine significant Digits: or it is a Number of Tens less than ten; which is therefore expressed by some Digit in the second place, and 0 in the first; or it is such a Number joined with a Number less than ten; which last Number is expressed by a Digit in the first place: or it is a Number of Ten-tens (or hundreds) less than ten; which is therefore expressed by some Digit in the third place, and 0 in the first and second places: But if with this Number is joined any lesser Number, (*i. e.* any of the preceding Classes) that is expressed by Digits in the first or second; or in both places. The Progression to greater Numbers, I think, will be now plain enough. Or we may consider it in this manner—Any Combination of Figures expresses some one determinate Number, according to the instituted Value of Places; and if Unity is joined to this Number, the Sum can be expressed: for this is done by changing the Figure in the first place, and taking the next greater, *i. e.* for 1, taking 2; for 2, taking 3, &c. and if 9 is already in the first place, then because nine and one make ten, we must (according to the Institution) put 0 in the first place, and change the Figure of the second place, taking the next greater; and if that is also 9, we put 0 also in the second place, and change the Figure of the third; and so on till we come to a place in which there is a Figure less than 9. So that if all the Figures of the given Example are 9's, we set 0 in all their places, and set 1 on the left of all. Example I. 47 and 1 is 48. Ex. II. 29 and 1 is 30. Ex. III. 499 and 1 is 500. Ex. IV. 999 and 1 is 1000.

Now it's plain, that if to any Number we can add one and express the Sum, by the same Rule we can add 1 to the Sum; and 1 to this last Sum, and so on. And because 1 may be the given Number, it follows that we can by this Institution express any Number in the natural Series from Unity *in infinitum*; and the way of doing it is here also made evident.

SCHOLIUMS.

I. We see now that tho' the Figure 0 signifies nought of itself, yet it is not useless; but indispensably necessary to fill up places, that other Figures may possess such places as they ought to be in, for the expression of certain Numbers which could not be expressed by this fundamental Rule, without the help of this Character. One Example is enough to shew it,

it, and we shall chuse the Number *Ten*, which could not otherways be express'd: for by this Institution, 1 in the second place is ten; but there cannot be a second place unless there be a first; and if any significant Figure stand in the first place, the whole will make a Number greater than Ten: so 14 is ten and four; therefore the Figure in the first place must signify *nothing*, and serves only to make two places, that the 1 may be put in the second, where it signifies Ten: thus, 10.

2. The Figure 0 standing on the Left-hand, or in the last place of any Rank of Figures is altogether useless; so 04 is no more than 4, because the Value of Figures rises from the Right-hand to the Left.

3. According to this Institution, each of the nine simple significant Figures may be consider'd as having two Values; the one certain and determinate, known by its Form, which is that it signifies simply by itself, (as 4 is *four*) and may be called its *simple* or *primitive* Value: the other is uncertain and variable, depending upon its place in a Rank with others; so any Figure in the second place is so many Tens; in the third place it is so many Ten-tens, (or hundreds) and so on in a ten-fold Increase: and this may be called the *secondary* or *local* Value, *i. e.* the Value of the *Place*; and these two Values compounded (or the one repeated as oft as the other contains Unity) makes the compleat Value of that Figure in that place. For Example, 4 in the second place is four-times Ten, (or forty;) and in the third place it is four-times Ten-tens, or four hundred. But in the first place (and in no other) these two Values coincide; for here there is no Value but the *Simple*.

4. If these two Systems of *Names* and *Figures* are duly compared, their mutual Correspondence will be easily understood; so that you'll find little difficulty in expressing any Number in the one manner which is first expressed in the other. But that no body may complain, I shall explain this more particularly in the following *Problem*.

P R O B L E M

Having any Number expressed by Figures, how to read or express it in Words; or, having it expressed in Words, how to write or express it in Figures.

CASE 1. *From a given Expression of any Number in Figures, how to read it in Words.*

We have already consider'd how that any Number being expressed by Figures, every Figure may be consider'd in a double view, *i. e.* according to its *simple* and *local* Value. Again, the System of Names as above explain'd, is so contriv'd, that Ten times Ten; Ten times Ten times Ten; Ten times Ten times Ten times Ten; [and so on, which are the Value of places after the second, which is Ten] have distinct Names, either simple or compound; and such Values taken any number of times less than Ten, are nam'd no other way than by expressing the number of times (less than Ten) that Value is repeated; so that if we know the Names of both values for every Figure in any Rank (or Expression of a Number, which has more than one place) and express each according to the Composition of these two Values, then we have the Expression sought, for the whole Rank or Number propos'd. The Names of the simple Values of Figures we have already learnt; what remains is to know the Names for the local Values, which is the Design of what they commonly call the Table of *Notation* (or *Numeration*) which I shall put in a more convenient Form.

T A B L E

TABLE of NAMES for the Local Values of Figures (in which X stands for Ten; C for Hundred, Th for Thousand.)

		Names	
th, Bill.	C	th, Bill.	C
th, Bill.	X	th, Bill.	X
th, Bill.	C	th, Bill.	C
th, Bill.	X	th, Bill.	X
Billions.	C	Billions.	C
th, Mill.	X	th, Mill.	X
th, Mill.	C	th, Mill.	C
th, Mill.	X	th, Mill.	X
Millions.	C	Millions.	C
Thousands.	X	Thousands.	X
Thousands.	C	Thousands.	C
Thousands.	X	Thousands.	X
Hundreds.	C	Hundreds.	C
Tens.	X	Tens.	X
Units.	C	Units.	C
18 th		18 th	
17 th		17 th	
16 th		16 th	
15 th		15 th	
14 th		14 th	
13 th		13 th	
12 th		12 th	
11 th		11 th	
10 th		10 th	
9 th		9 th	
8 th		8 th	
7 th		7 th	
6 th		6 th	
5 th		5 th	
4 th		4 th	
3 rd		3 rd	
2 ^d		2 ^d	
1 st		1 st	

If you would carry this Table farther, it is to be done by the Names *Trillions*, &c. (already explained) the same way used, as here *Millions* and *Billions* are.

Now the Use of the Table is plainly this. Apply each of these Names in Order, to the several Figures of a Rank, according to their Places (the Name Units signifying only, that the Figure in that Place is taken in its simple Value) and prefix the simple Value, making a Composition of both (observing how any simple Value compounded with Ten is contracted, viz. For two Tens, say Twenty, and so on to Thirty, Forty, &c. as above explained) and thus read all the Figures from the Left to the Right. Observe also, that where any Word belongs to the Names of several Places, which stand all next together, as do the Words, Thousand, and Million, and Billion, that needs not be repeated; but only once expressed, after all the Places to which it belongs are read according to what is more in the Name, as the following Examples will make plain.

EXAMPLES:

86, is Eighty Six

254, is Two Hundred, Fifty, and Four.

7408, is Seven Thousand, Four Hundred, and Eight.

67040, is Sixty and Seven Thousand, and Forty (*i. e.*) Sixty Thousand, Seven Thousand, and Forty.

46258300, is Forty Six Millions, Two Hundred and Fifty Eight Thousand, and Three Hundred, (*i. e.*) Forty Millions, Six Millions, Two Hundred Thousand, Fifty Thousand, and Eight Thousand, and Three Hundred.

24800540362, is Twenty Four Thousand and Eight Hundred Millions, Five Hundred and Forty Thousand, Three Hundred Sixty Two, (*i. e.*) Twenty Four Thousand Millions, and Eight Hundred Millions, &c.

But this may be made yet easier, by considering the several adjacent Places that have any one or more Words common in their Names; from which we have this other

TABLE.

th-Bill.	Bill.	th-Mill.	Mill.	Thoufd.	Un.
c : x : Un.	c : x : Un.	c : x : Un.	c : x : Un.	c : x : Un.	c : x : Un.

Observe, If you neglect the Word Un. (or Units) which stands first in the upper Line of this Table; and where the Word Unit stands after the three first Places in the under Line,

Line, put the Name that stands over every Repetition of $c : x : \text{Un.}$ repeating it also with the other two, *viz.* $c : x$. then you make the same Series of Names as in the preceding Table. And for the Advantage of this Form, we shall presently see it.

The Application of the TABLE.

Begin at the Right Hand of any Number, and divide or separate the whole Figures into Classes or Periods of three Figures each, as long as there are as many: then each Period must first be read as if it were alone by *Units, Tens, Hundreds, i. e.* the first place on the right of the Period by its simple Value, the second place as so many Tens, and the third place so many Hundreds as the simple Value expresses [which Practice is expressed by these Signs $c : x : \text{Un.}$ constantly repeated in the under Line of the Table.] Again, because Figures increase their Value towards the left, the same Figures in different Periods are of different Values, and therefore every Period has a common Name that compleats the Expression; which are these in the upper Line of the Table; (in which *Units* is made the Name of the first Period merely for a distinction, but is never expressed) therefore to apply this Table more easily, get the Names of the Periods by heart backwards and forwards; and then applying them to the Periods from the right to the left, that you may find the Name of the highest Period in any Example; and remembering exactly the Names of Periods in order from the left; you must begin from the left and first read the Figures of each Period, as $c : x : \text{Un.}$ then join the Name of the Period, which is supposed to be applied to each Figure of it.

Observe, That in some Cases you'll have a broken or incomplete Period of one or two Figures next the Left-hand, which must be read by itself, just as it happens to be; but the rest will each have three Figures: yet these may be also in a true sense but broken Periods, *i. e.* not have significant Figures in all their places, as in this 24,048. Again *Observe,* That in the Periods of Mill. and th-Mill. because the word Mill. is common, you need not repeat it: but in the th-Mill. use only the word Thousand, supposing the word Mill. which you need only express after the Figures in the Period of Mill. unless that Period be fill'd with o's, as here, 24,000,468,350. which is 24 th-Mill. &c. but in this, 24,360,579,200, I read 24 th. 360 Mill. &c.

See these few more Examples.

278,307,000	thus	278 Mill. 307 Th.
348,026,000,123	thus	348 Th. 26 Mill. 123.
7,200,809,867,345	read	7 Bill. 200 Th. 809 Mill. 867 Th. 345.
326,009,478,205,723	read	326 Bill. 9 Th. 478 Mill. 205 Th. 723.

CASE 2. *Any Number being expressed in Words, to write it down in Figures.*

If the preceding Table of Names corresponding to the several places of a Rank of Figures be consider'd, there can be no difficulty in this part of the Problem, which is but the reverse of the former; yet perhaps it may not be useless to some to point them out a Method, which is this—Remember exactly the Names and Order of the Periods from Left to Right; and of the three places in each Period: Then observe what is the Period first named in the Example, also what is the Number (*viz.* of $c : x : \text{Un.}$) applied to that Period; set down that Number, with a Point after it: then consider what is the Name of the next Period below that in order, and what Number is applied to it in the Example; set down that Number on the right of the former, with a Point after it; and whatever be the Number, it must be set down so as to possess three places, by setting o in those places of the Period to which no Number is applied; (except the first or highest Period where this is not necessary:) so that if there is no Number applied to any whole Period, (after that which is the highest in the Example,) set three o's in its

three places: thus proceed to the lowest Period or first on the right. *Observe* also, that if you find the Name *Thousand* mention'd before *Million* or *Billion*, it signifies *Th-Mill.* or *Th-Bill.* and is to be distinguished from the Period of simple *Mill.* or *Bill.* as in the preceding Table. One or two Examples will sufficiently illustrate this Practice.

Ex. 1. To write in Figures this Number; *Forty Million Two Hundred Thousand and Eight*; I proceed thus; *Million* is the highest Period; and *Forty* the Number applied, which I write down thus, 40; the next Period is *Thousand*, and the Number here applied is *two Hundred*, which I set on the right of the former, thus 40,200; then follows the Period of the *Units*, and the Number here applied is only *Eight*; which, according to the Rule, I set on the right of the former thus, 40,200,008.

Ex. 2. *Twenty-four thousand and sixty-two Millions.* Here the name *Thousand* standing before *Million* signifies *th-Mill.* which is the highest Period in the Example, and the Number applied is *Twenty-four*, thus written 24; then follows the Period of *Millions*, and the Number is *Sixty-two*, joined to the former thus, 24,062; and because there is no more in the Example, and yet there are two Periods remaining, *viz. Thousands* and *Units*, I fill them up with o's thus, 24,062,000,000. For without these, the other Figures could not express above *Thousands*. If you compare the Examples given for the first part reversely, you have enow for this last Purpose.

C O R O L L A R I E S to the FIGURAL NOTATION.

1. Of two Numbers, expressed by Figures, (by the Rule and Institution explain'd) that which has fewest Figures (or Places) is the least Number. *Examp.* 99 is less than 100. For tho' the Figures of that which has fewest be all the greatest possible, (*i. e.* all 9's) and those of the other be all the least possible, (*i. e.* 1 in the highest place, and the rest o's) yet the other will want at least 1 to make it equal to this; because 10 is equal to 9 and 1, and 100 equal to 99 and 1, and so on. Hence the value of an Unit in any place of a Rank is greater than the value of all the preceding Figures on the Right-hand; because taking that Unit in its true value, it is a Number having one place more than are upon its Right-hand.

2. If two Numbers expressed by Figures have an equal Number of places, that is the least Number which has the least Figure in the highest place; or in any other place, all the preceding places on the left being equal. *Ex.* 199 is less than 200; and 232 is less than 233; and 469 is less than 472.

3. If one Number is greater than another, and if you set before each of them an equal Number of whatever Figures, that which was greatest before will still be the greater.

4. If before, or on the Right-hand of any Rank of Figures, (or Number expressed by Figures) be placed any one Figure, the given Rank expresses thereby ten-times what it did before (or without that Figure;) and if two Figures are set before it, it expresses an hundred-times what it did before; and so on in a ten-fold Progression, according to the fundamental Rule. *Ex.* 480 is ten-times 48; and 4800 is 100 times 48, &c. And observe, That with respect to the raising the value of the given Rank, it is the same whether 0, or any other Figure be prefixed; it is certain that another Figure will make a greater Number of the whole Rank as it is now encreased; but will not make that part of it greater, which was the given Rank. So here in 486, the 48 is equal in value to 480, whatever Figure stands before it; tho' 486 is greater than 484 or 480.

5. Hence again; Any part of a Number taken from the Left-hand, may be valued thus; *viz.* we may consider what Number it makes taken by itself; and then consider the place in which its first Figure (on the right) stands; and make the Name of that place a common Name or local Value to the whole Rank: Thus, all the Figures of any Rank, excluding the first on the right, or place of *Units*, being read by themselves, (as if there were no Figure before them) is so many Tens; or such a Number taken 10 times. Again, excluding the two first places, the rest read by themselves is so many Hundreds, and so

fo on. Ex. 246873: here the 24687 is equal to 24687 Tens, (taking 7 as it were in the place of Units) *i. e.* twenty-four thousand six hundred and eighty-seven Tens. The 2468 is 2468 Hundreds; the 246 is 246 Thousands, and the 24 is 24 Ten-thousands: as the 2 is 2 Hundred Thousands.

SCHOLIUMS.

1. Both in the General Rule of the Figural Notation, and these two last Corollaries, I have supposed this for a Truth, *viz.* That Ten-times, (or a Hundred, &c.) any Number is the same as that Number of times Ten, (or a Hundred, &c.) For Example, that Ten-times 7, is 7 times Ten; which is a Truth I believe will be easily granted: yet it is of that kind that admits a Demonstration; but I shall refer it till we come to *Multiplication*, to which it properly belongs.

2. By the Figural Notation now explained, all Numbers above Nine are expressed in a compound Form; representing either a certain Composition of the Number Ten, or such a Composition with the addition of a Number less than Ten; which are notably distinguished by this method of Expression. But there are various Degrees of the Compositions of Ten; for a Number compounded of Tens is either a Number of Tens less than Ten, which is expressed by a Digit in the second place and 0 in the first, as 10, 20, &c. or it is such a Number taken ten-times, which is expressed by a Digit in the third place, and 0's in the second and first, as 300, 400, &c. or such a Number as the last taken ten-times, as 2000, 5000, &c. and so on. All which different Degrees are express'd by a Digit in different places. From which it's clear, that if several of these degrees be joined together, *i. e.* if there is a Rank consisting of more than one significant Figure, and having at least one 0 on the Right-hand, as 460, which is 46 Tens, that expresses a Number which is also a Composition of Tens; but if there is a significant Figure in the place of Units, that Number contains so much odd or over a certain number of Tens, as 468, which is 46 Tens and Eight. Now there are two things to be remarked from this Method of Notation: First, That every Number is distinguished by the very Expression into as many parts as there are significant Figures in it; each of which is expressed separately by setting as many 0's before it as there are Figures before it in the given Expression; as in the following Example. And secondly, That each of these Parts is a certain Composition of the Number Ten (as above explained) except that which is in the place of Units, which is always less than Ten. Thus, the Number 4682 is equal to 4000 and 600 and 80 and 2, which is more simply and conveniently written all in one Rank 4682; whereby all the Parts are as clearly and intelligibly marked out by the meer situation of the Figures, (according to the Rule.) Now tho' this is really a compound Form, expressing several Numbers distinctly from one another; yet because it is the most simple way of expressing that Number, or Term in the natural progression, which is equal to the Sum of all these lesser Numbers (according to the Institution) therefore it is said to be a simple Expression of one Number, in comparison of the other ways of expressing the Parts separately; or of the Expression of a Number by any other of its component Parts, separately written each in their most simple Form, (in each of which therefore there are Parts, or Figures that have the same local Value) as if instead of 8 we should write 5 more 3, (which together make 8) or instead of 74 we should write 43 more 31, which therefore in distinction from the other we may call complex Expressions representing separately two different Numbers, the lesser of which has Parts of the same local Value with the greater. Or we may explain this distinction of simple and complex Expressions, thus: That Expression is called simple, or one Number, which is one of those whereby the natural Series of Numbers is expressed in a continued Progression from Unity; but when two different Expressions of that Series are separately written in two distinct Ranks, they are said to be two different Numbers: and with respect to that Number to which they are both together equal, they are said to be a complex Expression

of it; especially with some Mark or Word betwixt them to signify their being joined together. For Example, 46 is a simple Expression of one distinct Number in the natural Series: But 32 and 14 are two Numbers, which being both together equal to 46, therefore 32 more 14, is a complex Expression of the Number 46. Again, in the same manner any Number which is equal to the difference of two Numbers may be complexly expressed by these two, with a Mark of Subtraction betwixt them: As for Example, since 32 more 14 is equal to 46, therefore if 14 is taken out of 46, the remainder is 32; and hence 32 may be expressed thus, 46 less 14.

The same things are applicable to the Parts by which any Number in its simple Form is expressed; as if for 46 we should write 40 more 6; or for 6 we should write 46 less 40.

In the last place observe, That the compound Form of this Notation is the Perfection of it; not merely as it is compound, but the particular Manner of it; because of the certain relation that each part has to one Number, *viz.* Ten, by the constant, regular and uniform Progression in the Value of Places: For by this Similarity or Likeness of some Parts in different Numbers, (*i. e.* of Parts expressed by Figures in the same or like places;) and the Connection or Relation of all their Parts by means of the common Number Ten, to which they have all a relation; Numbers can be compared together, and their Operations performed in a very easy, clear and distinct manner. And more particularly we have from the Method of Notation this Principle of Operation, *viz.* That some Operations may be performed by the few simple Characters, taken and applied in their primitive Values; and what is wrong or deficient by the neglect of their local Values may be again made up, by the order or due placing the several Figures arising from the Operation: And in other cases where we cannot work by single Figures, yet we can take the Parts of Numbers, (*i. e.* two or more of their Figures) and consider them in the Value they would have by themselves; correcting the Defects the same way as now mentioned. Which Principle we shall see applied in the following Rules; and here only I shall further observe, That in this it is that the Rule of Notation is the ground of all other Operations; affording regular and easy Rules for expeditious and certain Work, as we shall learn.

As to the History and Invention of this admirable NOTATION, see what is said in the *PREFACE*.

3. We might here make Comparisons betwixt this Method of Notation, and others that have been or may be used, in order to shew the Excellency of this Method; but this will be better understood after you see the Application of it in the Operations, where I have allotted a Place for some general Reflections upon these Operations. However, I shall here explain the *Notation of Numbers* instituted by the *Romans*; which will be proper, because it is also practised for Marking of Chapters and Sections, and such things; and this being compared with what we use, the difference in favour of our Method will be sufficiently evident.

Of the ROMAN NOTATION:

The Characters whereby the *Romans* marked Numbers were taken out of their Alphabet of Capital Letters; thus:

<i>The Simple Characters.</i>							
	I.	V.	X.	L.	C.	D.	M.
Equal to	1.	5.	10.	50.	100.	500.	1000.

The

The intermediate Numbers betwixt these are expressed by a repetition of the same; setting them together in a Line, thereby expressing the Sum of all their Values (above expressed) joined to one another, (for they have no different Values from their Places) the Characters of greatest Value being set next the Left, as for Example; II is 2. III is 3. VI is 6. VII is 7. XI is 11. XV is 15. XX is 20. LX is 60. LXV is 65. DX is 510. DC is 600. DCCCC is 900. DCCCCLXXXVIII is 999. These Examples will sufficiently shew how all the rest are made.

But to prevent too great a Repetition of the same Characters, they sometimes set the lesser Character before the greater; and then it represented the difference of these two, or the effect of taking the one Number away from the other; thus IV is 4. IIX is 8. IX is 9. XL is 40. CD is 400. CM is 900. And when a Number is expressed by more than two Characters, if any part of it is thus expressed, it is fit to distinguish it from the Characters on the Left of it, by a Point; thus, 140 may be expressed C_.XL, (for CXXXX) and 148 thus, C_.XLIIIX (for CXXXXVIII.) Again, 499 thus CD_.XCIX, instead of (CCCCCLXXXVIII) which are more convenient.

Again, for Numbers greater than 1000 or M, they are expressed after the same manner. But there are other things in their System, both for some Numbers less than 1000, and especially for greater, which I shall briefly explain. Thus, for D or 500 they write ID. and then by adding another I it gradually expresses ten-times as much; so IDD is 5000. IDDD is 50000, and so on. Again for M or 1000 they write CID, and by joining another such Mark as C and I one on each hand, it expresses ten-times as much; so CCIDD is 10000. CCCIDDD is 100000. But lastly, they had a more convenient way of expressing any Number of Thousands, which was by drawing a Line over any Expression of a Number less than a Thousand; whereby it expressed so many Thousands: so \overline{V} is 5000. \overline{VI} is 6000. \overline{X} is 10000. \overline{LX} is 60000. \overline{C} is 100000. and \overline{M} is a Thousand-thousands, or a Million, 1000000. \overline{MVI} is 2000000. But I shall insist no more.

If we now compare this Method of Notation with ours, it presently appears by the preceding Examples, that some Numbers are more shortly expressed by the *Roman Way*; but these are very few in respect of what are otherwise: And then there is here no such regular Progression in the Value of the same simple Characters as in the other Method. But we must learn the worth of this from its Application.

Of the UNIVERSAL NOTATION.

By this Name is not meant any constant or established Method used every where for the Expression of particular Numbers; but a Method of representing any Number indefinitely, in order to the more easy and general Expression and Demonstration of certain Truths in Numbers, which tho' they be limited to particular Conditions, yet not to particular Numbers, but extend to all Numbers, wherein the same Conditions are found.

The fundamental Principle of this Notation is the same as the last, *viz.* That any Mark or Sign may be instituted for the Representation of any of our Ideas; and here it is done by Letters, making the same Letter stand indifferently for any Number, upon this Condition, That through the same Proposition and Demonstration it be supposed to stand for the same Number; *i. e.* when particular Examples are applied, we must apply the same Number always to the same Letter. Thus the Letter A or B, or any other, may represent any Number we please to suppose.

Observe, This Notation, and the consequent Operations are called *The LITERAL* or *SPECIOUS Arithmetick*, which is in part the Foundation of the *Algebraick Art*; the designed Use of which in the following Work is to make easy and universal Demonstrations.

For when any Truth is proposed which is not limited to particular Numbers, but only to certain Conditions; it is not a sufficient Demonstration to shew that it holds in one, or any

any Number of particular Examples; it must be shewn that it will hold good in all Cases possible: and as this must be done by an universal Method of Reasoning abstracted from all particular Examples; so there must necessarily be an universal Notation for Numbers. It is true indeed, that the Universality of a Truth may, in some Propositions, be made to appear through one, or a few particular Examples; but for the most part this would prove very tedious, and require many Words, which would render the Demonstration more difficult and obscure; and in very many Cases could not be done at all. Which makes the *Algebraick* Method of Demonstration necessary in Arithmetick, (as I have more fully represented in the *Preface*.) The Principles and Rules of which, as far as this Work requires, you'll find explained gradually as we proceed.

Before we enter particularly upon the other Rules and Operations of Arithmetick, there is another Distinction of Numbers must first be explained; *viz.*

§. 2. *Of Numbers INTEGRAL and FRACTIONAL.*

THIS Distinction proceeds from the Comparison of lesser Quantities with greater; and to understand the Nature of it aright, we must consider the different Notions of *Parts* and *Whole*; thus, Every lesser Quantity is called a Part with respect to a greater of the same Species, which is called an *Integer* or *Whole* with respect to the lesser.

But there is a more general, and also a particular Sense in which one Quantity is called a Part of another. In the general, no more is meant but that it is a lesser Quantity, which with some one or more Quantities, also lesser, make up a Quantity equal to that other; and into which therefore that Greater may be resolved. But in a more particular Sense, a Part signifies such a lesser Quantity as is contained a number of times precisely in a greater, (or Whole:) *i. e.* it is one of those lesser Quantities, all equal among themselves, into which any Quantity may be resolved or separated. Or we may also conceive it thus, *viz.* As a lesser Quantity, of which a certain Number joined together makes up a greater Quantity or Whole. Hence such a Part is called an Equal or *Aliquot* Part; and the number of times it is contained in the Whole, or the Number of equal Parts contained in the Whole, gives a Denomination to the Part, and is called its *Denominator*: so if any Quantity is contained in a greater 6 times, it is called a sixth part of it. Now it is in this sense only that a Part can be understood in Arithmetick: for in order to compare two different Quantities together by the means of Numbers, we must consider them as composed of some common Element, or equal Part; by the Number of which contained in each, the Comparison may be made; and the Value of these Quantities with respect to one another be determined. If the lesser is an *Aliquot Part* of the greater, there is no more to be done; but otherwise they must both be conceived as composed of, and reducible into, some common Element or equal Part; so that if the lesser is not one of these *Aliquot* Parts of the greater, yet it is equal to a greater Number of such Parts; [for otherwise they cannot be compared together; at least the Relation cannot be expressed in Numbers.] Such a lesser Quantity, to distinguish it from an *Aliquot* Part, is called an *Aliquant* Part. For Example, a lesser Quantity equal to 2 of 3 Parts of another, is called an *Aliquant* Part; thereof.

I have hitherto spoken of Quantities and their Parts in general; but what is said is applicable both to what is called *Continued* Quantity, (as Length, Weight, Time, &c.) or to pure Number. For every lesser Number is a part of a greater; and is either an *Aliquot* or *Aliquant* Part, because Unity is the common Element or *Aliquot* Part of all Numbers; so that every Number is a Number of such *Aliquot* Parts (as Units are) of every other Number. But this difference is very remarkable, That the *Aliquot* Part of a continued Quantity, considered properly by itself in the nature of a continued Quantity,

is only one single individual thing, or an *Unit*; whereas in pure Numbers, one *Aliquot Part* may be a Number greater than Unity. *Ex.* The 3^d part of 12 is 4 (for 3 times 4 is 12.) It is true indeed, that conceiving any continued Quantity to be resolved into 12 equal Parts, one 4th part of it is equal to 3 of these parts; yet the Whole and Part are here consider'd only as pure Numbers: for to say 3 is a 4th part of 12, whatever things we speak of, it is but a pure Arithmetical Expression; whereas in continued Quantities, because the Part of any thing must be of the same nature with the Whole, therefore a 4th of any Length (for *Ex.*) must be one certain Length; which, in so far as is necessary to constitute a 4th part, is not conceived to be any further divided, but to be an entire Length equal to a 4th part of another. But a more remarkable thing is, that any continued Quantity may have any *aliquot Part*, for that we can conceive at pleasure; but pure Numbers cannot: For some have no other *aliquot Part* but Unity; as 5 and 7; and others have different parts according to their Compositions: so 6 has a half and a third part, but not a 4th part or a 5th part. Again, no Number can have such a Part as is denominat'd by a Number greater than itself; for Unity is the least part of any Number, and is the part denominat'd by that Number itself; so 3 has not a 4th part in pure Numbers: But if we consider any Number applicately as signifying a Number of things which are divisible into any conceivable Parts, then any *aliquot Parts* of one, or of any number of these things is possible. For *Ex.* Tho' a 4th part of 3 is impossible in pure Numbers, it is possible when the 3 is applied to things divisible into any Number of Parts as 4; yet here it must be carefully remarked, that this is not the 4th part of the Number 3, but of a greater Number into which 3 things are resolved: Therefore every such Expression as a 4th of 3, or 3 4^{ths} of 2, must be conceived with this qualification, *i. e.* as possible only in *applicate Numbers*.

The same also is to be understood, tho' the Denominator is less than the Numerator which is considered as the *Whole*, when this Number has no such Part as is expressed; as in this *Ex.* a 3^d of 5.

We shall now gather together these Definitions; and from them you'll see the ground of the Distinction propos'd with the Definition of the Terms.

DEFINITIONS:

1. Every lesser Quantity or Number is in a more general Sense, a Part of every greater (of the same Kind,) which is called a *Whole* or *Integer* with respect to the *Part*. But more particularly,

2. An *Aliquot Part* is that which is contained a certain number of times precisely in the Whole; and that number of times is the Name or Denominator of the Part. *Ex.* If any Quantity or Number is resolved into 3 equal Parts, one of them is an *aliquot Part*, called a 3^d part; so 2 is a 3^d part of 6. And from the nature of Numbers we have this *Corollary*, *viz.*

COROLLARY. One is an aliquot Part of every Number, and the Denominator is that Number itself. So 1 is the 6th part of 6.

3. An *Aliquant Part* is such a lesser Quantity or Number which is not an equal Part; but contains a certain Number of some equal (or *aliquot*) Parts of the Whole. As when any Quantity or Number is not one *aliquot Part* of a greater, yet is equal to 3 4th parts of that greater: So 9 is 3 4th parts of 12; for a 4th of 12 is 3, and 3 times 3 is 9.

4. An *Integral* (or whole Number) is that which represents things absolutely by themselves, without any comparison to other things; *i. e.* they are not considered as Parts of other things; as when we say in general 4 Things, or particularly 4 Words: And they are called *Integral* or whole Numbers only in distinction from Parts.

5. A *Fractional Number* (or a *Fraction*) is that of which each Unit represents a certain *aliquot Part* of another thing, as the Whole to which this Part relates, called hence the *Relative Integer*. For *Ex.* 1 5th or 3 5th parts; or 7 13th parts of any thing. And because the Denomination of the Part, which is also a Number, must be expressed, therefore every Fraction consists of two Members, or requires two Numbers: for there is the
Number

Number of things directly and immediately represented (as in the preceding *Examples*, the Numbers 1, 3, 7.) called hence the *Numerator* of the Fraction; and the Number of equal Parts of which the Relative Integer is supposed to be composed, called the *Denominator*, or the Name of the Part, expressing the Value of each Unit of the Numerator with respect to the Quantity of the Relative Integer. So in this *Ex.* 3 5th parts, 3 is the Numerator, and 5 the Denominator; the common way of placing them being to set the Numerator above the other thus, $\frac{3}{5}$ or $\frac{3}{5}$ Numerator.
Denominator.

SCHOLIUM. From this Definition of a Fraction it is plain that the Numerator may either be less or greater than the Denominator, or equal to it; for we may as reasonably say $\frac{7}{5}$ (or 7 5th parts) as $\frac{4}{5}$; if we understand it according to the Definition, *i.e.* as expressing 7 things each of which is equal to a 5th of another thing; and not as if 7 were supposed to be taken out of 5, which is impossible. By comparing the Numerator and Denominator, we have this Consequence; *viz.*

COROLLARY. If the Numerator is greater, equal to, or lesser than the Denominator, the Quantity expressed by that fractional Number is greater, equal to, or lesser than the Integer; because the Denominator represents all the Parts of the Integer, and the Numerator shews how many are taken.

And this gives rise to a Distinction of *Proper* and *Improper Fractions*, as the Numerator is less or not less than the Denominator. But a more particular Explication of this, with other Distinctions of Fractions, we must refer to another place. And here you may *Observe*, That instead of the Names *Integral* and *Fractional*, we might as properly call them *Absolute* and *Relative* Numbers; which do very well express their different Natures: For the first considers things simply and absolutely in themselves; and the other considers things relatively, as Parts of other things.

It is to be also *Observed*, That Fractions are a more general Kind of *applicate* Numbers: For the Numerator (or the Number of things directly designed) is restrained; so that it does not represent a Number of any things indifferently; but is limited to a certain Relation to some other thing: nor does it express any Part of that other thing; but such a Part or Parts as the Denominator expresses: yet while there is no particular thing named as the relative Whole, it is in this respect a general and *abstract* Fraction, (but not a pure absolute Number;) so $\frac{2}{3}$ is a general and *abstract* Fraction; but $\frac{2}{3}$ of a Day is *applicate*. Wherefore in every *applicate* Fraction there are two Denominations to be considered, which we may call the *Relative* and the *Absolute*: The first is the *Denominator* of the Fraction, and the other is the Name of the Integer. But if the Integer is not one, but a Multitude of Things (as $\frac{2}{3}$ of 6 Pounds) that is to be conceived as an Integer or *one* of its own Kind; or rather we are to conceive this Expression as a mixt Form reducible to a simple, wherein the Integer is an Unit of a particular Name; so $\frac{2}{3}$ of 6 is equal to $\frac{4}{3}$ of 1. But this must be left to its own place.

GENERAL SCHOLIUM.

Of the different Senses in which Unity is Divisible and Indivisible, and the Conversion of Numbers from Integral to Fractional, and contrarily; shewing in general wherein their Operations must be the same, and wherein they differ.

Here now is the place to explain the different Senses in which Unity is divisible or indivisible. And in the first place this is plain, That *Unity* in its own Nature as Number is *indivisible*; for there can be no Number of Things conceived less than *One*: but if we consider the Subject or Thing to which the Idea of *Unity* is applied; as that is capable of division into real or imaginary Parts; or as it is really a Collection of distinct Things united by the Imagination; so, what is the Subject of *Unity* in one View and under one Denomination, may be the Subject of Multitude under another. For *Ex.* 1 Pound is the same as 20 Shillings; wherefore it is not the Number *One* that is divisible, but some

continued Quantity, as a Yard, a Day, &c. or some Number of Things comprehended under a singular Denomination. Hence again we learn to correct a Vulgar Error, *viz.* That a Proper Fraction (*i. e.* whose Numerator is less than its Denominator) is a Number less than *Unity*. It does indeed represent a Quantity less than the relative Integer or Unit; but is not a Number less than *Unity*: For the *Numerator*, which cannot be less than 1, is as properly a Number as if it were applied to things under an absolute Denomination: so $\frac{3}{5}$ of a Pound does as truly express three things, as 3 Pound does; differing only in the Value and Way of denominating the things. Again, because one Quantity or Number cannot be referred to another as a Part or Parts, unless that other be really or conceivably divided into, or composed of such a Number of Parts; therefore, strictly speaking, the relative Integer of every Fraction is what we may call a *collective Unit*, or a real Multitude united together in one Whole under a particular Denomination. Hence a Proper Fraction is in effect some lesser Number compared to a Number greater than it, and always greater than *Unity*. For *Ex.* $\frac{3}{5}$ is 3 things taken out of 5; or 3^{5th} Parts of a Whole composed of 5 Parts: and in this View only a Proper Fraction is a lesser Number than its relative Integer; yet not as this is *Unity*, but as it is really a Multitude. To have done; The proper Arithmetical Value of One or any other Number is invariable, *i. e.* One, Two, &c. is always the same Number, in whatever things and however denominated; but take the Number with the Application complexly, there may be a difference; so that what is equal to *Unity*, or any other Number, in one denomination, may be a greater Number applied to another; as 1 Shilling is equal to 3 Groats, or to 12 Pence. The mixt Value is the same in all these, yet the Numbers and Denominations differ: Also what is an Integral Number applied to Things under a proper and absolute Denomination, may be converted to a Fraction or relative Number by applying a relative Name or Denomination; so 3 Shillings is the same as $\frac{3}{20}$ Parts of a Pound. In both Expressions the Number and mixt Value is the same, only the Things are differently denominated in the Application; this being indeed all the difference betwixt Numbers *Integral* and *Fractional*; yet this difference is the occasion that these two Kinds must be handled separately: for the Denominator of a Fraction being also a Number, respect must be had to that in every Operation, which occasions more Work than in Integrals. But still the fundamental Operations of Numbers are those performed by Integral or Absolute Numbers; for the Numerator and Denominator of a Fraction, taken by themselves, are of the same general Nature with every other absolute Number, and can have no other Operation applied to them; and the way of making that Application so as to fulfil all that both the Denominations, Relative and Absolute, (where both are considered) do demand, is the only new thing in the Operations of Fractions. Therefore after the Operations in Whole (or Absolute) Numbers are explained, which will employ the remaining Chapters of this Book; the same shall be done for Fractions in the second Book.

C H A P. III.

ADDITION of *Whole and Abstract* NUMBERS.

D E F I N I T I O N.

ADDITION is the finding one Number equal to two or more Numbers taken all together; that is, finding the most simple Expression of a Number, (according to the established Notation) containing as many Units as are in all the given Numbers taken together; which is hence called their *Sum*. For *Example*, the *Sum* of these Numbers 8, 17, 24, 675, is found by the following Rule to be 724.

SCHO-

SCHOLIUMS.

1. Before we enter upon the particular Rules of *Addition*, it is necessary to make the following Reflection upon the Method of *Notation*, as it is in part the Foundation of all other Rules of Operation; *viz.* That as by any established Notation, whatever it be, we know how to express any Number, or the whole Series of Numbers in a continued Succession from *Unity*, by the adding or joining Unity after Unity for ever; so by Application of this Institution, and the general Axiom, that *The Whole is equal to all its Parts*, we see a possible way of finding and demonstrating the Sum of any two Numbers, *viz.* by beginning with one of the given Numbers, (and it is best to take the greater) and joining to it all the Units of the other one after another, expressing the Sum at every step, according to the Rule of *Notation*, till the last Unit is added; and then you have the Sum sought: because if all the Units in any Number are added successively to another Number, the first Number (which is nothing else but all its Parts together) is certainly added to the second Number. For *Example*, If (according to

the present *Notation*) it were proposed to find the Sum of 8 and 7: I take 8, and after it I set down all the Units of 7 separately, and by adding them one by one to 8, and expressing the Sum as it gradually increases, the last and total Sum is 15. In the same manner may we find the Sum of any other two Numbers, or of any Numbers more than two, by first adding any two of them, and then to the Sum adding any other of them; and so on.

Let us next observe, What is arbitrary, and what is not so in this Operation. In the first place, As the whole System of the Signs of Numbers, both by *Words* and *Figures*, is a pure arbitrary Institution, so the Addition or Expression of the Sum of Unity, and any Number, is directly and immediately a Part of that Institution; and therefore has no other Reason or Demonstration but that it is so Instituted. Again, tho' the Names of all Numbers are contained in the Institution, yet the finding the Sum of any other two Numbers, or calling it by such a Name in the System, is not immediately a Part of the Institution, but a Consequence, and therefore Demonstrable: For the pure and simple Institution is all comprehended in the System of Signs taken in a gradual Succession from Unity, and proceeding by a continual Increase of Unity; and therefore contains immediately no Question or Case of Addition, but that of adding Unity to Unity, or to any other Number; all other Sums being found by Consequence from this, which therefore have a proper Demonstration, different according as the Consequence is less or more remote. As you'll afterwards learn.

Now the Method of Addition most immediately connected with the Institution is that above explained: But it is easy to perceive how tedious and insupportable this Method of Addition would be upon any System of Notation; and as upon different Systems, the Remedy of this Difficulty would be less or more perfect, so the present admirable Method of *Notation* affords the most easy and perfect Rules for *Addition*, (and all other Operations) whereby such Additions are performed by a few and easy steps, which cannot be done all at once, (as we add Unity to any Number) and would be insupportably tedious to do by so many steps as the preceding Method prescribes: yet this is to be observed, that as the established *Notation* is the Ground-work and Foundation of all, so there are some simple Cases that can be done no other way; as shall be presently explained,

The most simple Cases in any *Problem* are first in the Order of Science; and here the Addition of Unity to any Number is the first and most simple Case: but as it is contained immediately in the Rule of *Notation*, therefore it is supposed in the following *Problem*, as a previous and fundamental Principle.

2. This Sign or Character $+$ set betwixt two Numbers, signifies the Addition of the one to the other; and is a complex or indefinite way of representing the Sum: thus, $3 + 4$ signifies that 3 and 4 are added together; and we read the Sign by the word *more*. *Exam.* $3 + 4$ is 3 more 4; and thus it expresses the Sum in a complex manner by the

the Parts. And when more Numbers are added, they are joined by the same Sign; thus $3 + 4 + 9$ is the Addition or Sum of 3, 4, and 9. The Use of this in particular and determinate Numbers, is chiefly by way of Abbreviation for the neater and shorter Explanation of the Work of Addition in particular Examples, as you'll see immediately.

But the principal Use of this Sign of Addition, is for the Expression of the Sum of Numbers, universally or indefinitely represented by Letters in the Algebraick Art. Thus, Any two Numbers being represented by A, B, their Sum is expressed indefinitely thus, $A + B$, and the Sum of A, B and C, is $A + B + C$, and so on; which is the General Rule of the Literal Addition, or Numbers expressed by Letters. Some other particular Considerations relating to this, you'll find in another Place.

Observe again, That as the same Number may be variously represented, either by one simple Expression, or by the simple Expressions of other Numbers variously applied to one another by sundry Operations, whose final Result brings out the same Number; so to express the equality of Value betwixt these different Expressions of the same Number, we use this Sign $=$ set betwixt them; which we use for Abbreviation in explaining the Work of particular Examples in all the common Operations. Thus 8 added to 7 makes 15, which is therefore expressed either complexly $8 + 7$, or simply 15; and to signify the Equality of these Expressions, we write $8 + 7 = 15$, and read it thus, 8 more 7 is equal to 15. Universally, $A + B = D$, signifies that the two Numbers expressed by A and B, are together equal to the Number expressed by D; and so of other Examples of Addition. As $3 + 4 + 5 = 12$, or $A + B + C = D$. Applications of this, in other Operations, you'll find in their Places. I have only this further to say here, That the different Expressions of the same Number, constitute what in the Algebraick Art is called an *Equation*, that is plainly an Equality of Value betwixt two Expressions of Number: In the finding of which, from the Conditions and Circumstances of any Proposition, with the various Changes and Transformations to be made upon them, by the Application of different Operations, whereby one Equation is deduced from another, consists the Algebraick Art of Reasoning; which, so far as the present Undertaking requires, you'll learn as we proceed.

P R O B L E M.

To add two or more Numbers into one Sum, or simple Expression.

CASE I. *To add any two Digits or Numbers less than Ten.*

Rule. Take the greater of the two, and to it add all the Units of the other one by one; expressing the Sums gradually according to the Rule of Notation, as explained in *Schol. I.*

Exam. To add 9 and 6, it is, $9 + 1 + 1 + 1 + 1 + 1 + 1 = 15$; for adding the 6 Units gradually, and expressing the Sums in the Order of Notation, they make this Series, 9, 10, 11, 12, 13, 14, 15, the last whereof is the Sum sought.

Demon. The Reason of this Rule is already explained; and that there cannot be another Way of adding two Digits, is evident.

SCHOLIUM. Practice by degrees fixes the Sums of all the various Examples of this Case in our Memory; whereby we become capable to pronounce the Answer as readily as the Question is proposed: For upon Reflection it will be found, That we do not always calculate, and add in the manner directed; but know the Sum purely by Memory; which was no doubt acquired by repeated Practice of adding them together Unit by Unit; for it could be done no other way: and therefore it is the only Method we can take to teach young ones, who know nothing but the Names of Numbers in the simple progressive Order from Unity; who may be assisted by their Fingers in this manner, *viz.* Let them tell out the Units of the lesser Number upon their Fingers, then take the greater Number and add to it the Units of the other from their Fingers, expressing the Sums gradually according

according to the progression of Names, and the last is the Name of the Number sought. Again, if thro' any confusion of Thought, the Sum of two Digits should not readily occur, or if one should pretend to deny or doubt of it, this is the natural and certain way of finding or demonstrating it; [which may also be done by dissolving one of the Numbers into two Parts, and adding them successively to the other Number, which is only reducing the Question to a more simple Case, where the Sum may be more easily remembered.] And this brings to my mind, That in Conversation I have met with Persons who would affirm that the Addition of two Digits is a thing not properly demonstrable, but the immediate Effect of an Institution; the contrary of which, I think, I have sufficiently shewn. The Occasion of this Opinion may be the familiarity we have with these simple Cases from our first acquaintance with Numbers; so that remembering the Sums as readily as we do the Sum of Unity and any Number, we are apt to fancy we came by them without any Reasoning or Calculation, because we have them so now. I must also observe, That the Method of our common Books of Arithmetick may have contributed to this; for in these we have but one general Case and Rule of Addition, in which it is supposed that we know already, or can find the Sum of any two Digits; and this perhaps is done upon a Supposition of its being simple and easy: But this would be no reason for omitting it in a Work designed for a just and rational System; which must therefore explain the Connection and Dependence of all the Parts of the Science upon their first Principles, and upon one another.

In the last place, then, since this Case is supposed in all other Cases of Addition, it is necessary the Learner know the Sum of all its Examples as readily as the Question is proposed. In order to which; I shall express them all in the following Table; from which they may be got by heart more easily by those who are Beginners in this Science.

TABLE shewing the Sums of any two Digits.

1	2	3	4	5	6	7	8	9	0
2	3	4	5	6	7	8	9	10	1
4	5	6	7	8	9	10	11	2	
6	7	8	9	10	11	12	3		
8	9	10	11	12	13	4			
10	11	12	13	14	5				
12	13	14	15	6					
14	15	16	7						
16	17	8							
18	9								

The Construction of the Table is obvious, and the Manner of using it is this: Take the greater of the two Digits, whose Sum is sought, in the upper Line, and the lesser on the Right-hand Column; in the same Line with this, and under the other stands the Sum. So under 8, on the Head and in the same Line with 6, on the Side stands the Sum 14.

CASE 2. To add any two or more Numbers into one Sum.

Rule 1. Place the given Numbers under one another, so that the Figures in like places of each be directly in one Column, [i. e. all Figures in the first or Units place in one Column; all in the second place, or Tens, be also in one Column; and so on in the Order of Places.] Then *secondly*, begin at the Units place of the lowest Line or Number; add that Digit to the next above of the same Place and Column; and to this Sum add the next Digit, and so on till all the Digits in that Column are added; then if the Sum is less than 10, set it down under the Figures added; and add up the second Column the same way; and so on thro' all the Columns, taking the Figures in each according to their simple Value. But if the Sum of the first, or any other Column exceeds 10, then it is either a precise Number of Tens (as 20, 30, &c.) or it is such a Number of Tens with some Number.

Number less than 10, (as $24 = 20 + 4$; or $68 = 60 + 8$.) In the first Case write down 0, and in the other the Number over a precise Number of Tens, (as 4, if it is 34; or 8, if it is 68) and for every 10 in the Sum carry 1 to the next Column; *i. e.* add the Number of Tens in the Sum of every Column to the next Column. Having thus gone thro' all the Columns, the Number of Tens in the Sum of the last Column is to be set down on the left of all the Figures already found; or, the whole Sum of the last Column is set on the left of all the preceding Figures: and all these Figures thus found and placed, express the Sum sought.

$$\begin{array}{r} 654 \\ 243 \\ \hline 897 \end{array} \text{ Sum.}$$

Ex. 1. $654 + 243 = 897$. wrought as in the Margin thus: Beginning at the Units place of the lower Number, I say, $3 + 4 = 7$, which I set under the Numbers added; then $4 + 5 = 9$; and lastly, $2 + 6 = 8$; and the total Sum is 897.

$$\begin{array}{r} 897 \\ 968 \\ \hline 1865 \end{array} \text{ Sum.}$$

Ex. 2. $897 + 968 = 1865$. Thus, $8 + 7 = 15 = 10 + 5$; therefore I set down 5, and carry 1 to the next place; then $1 + 6 = 7$, and $7 + 9 = 16 = 10 + 6$, for which I set down 6, and carry 1 to the next place; then $1 + 9 = 10$, and $10 + 8 = 18$, which being set down, the total Sum is 1865.

Let the Learner practise more Examples of this Kind to himself.

SCHOLIUMS.

1. If there are more than two Numbers to be added, this Rule plainly supposes, that we can readily, in our Mind, add any Digit to any other Number; which therefore might have been considered as a particular Case: But, rather than make too many Cases, I have made this Supposition; nor has it any thing contrary to good Order, because it is indeed no other than a particular Case comprehended under the general Rule here delivered: so that by practising Examples of this simple Case, we soon acquire the Capacity supposed for more complex Cases. For, if the greater Number has, in the place of Units, such a Digit, as added to the other Digit, makes a Sum less than 10, the Total is plain; thus, $64 + 2 = 66$; $22 + 7 = 29$. But if the Sum of the two Digits exceeds 10, what's over belongs to the place of Units of the Sum, and 1 for the 10 is to be added to the remaining part of the greater Number. Thus $46 + 8 = 54$, *viz.* $6 + 8 = 14$; which makes 4 in the place of Units of the total Sum, and then $1 + 4 = 5$ in the place of Tens. Also $397 + 9 = 406$ for $7 + 9 = 16$, then $1 + 39 = 40$.

There is also another Supposition in this Rule, *viz.* That we can readily perceive how many Tens are contained in any Number, or in the Sum of any Column of Digits, and also what remains over. Now, tho' these do indeed belong to other Rules, yet the Rule of Notation has already taught us them, in the Cases here supposed: For, if we write down any Number, the Figure in the place of Units is the Number over all the Tens contained in it, and the remaining Figures on the Left-hand, taken by themselves, express the Number of Tens; as has been explain'd in the Rule of Notation (see Corol. 4. after the Problem in Chap. 2.) thus, 40 is 4 Tens and 0 over, 87 is 8 Tens and 7 over, 124 is 12 Tens and 4 over. In small Examples the Number of Tens is easily perceived without writing down the Sum, and in greater Numbers write it down. But there are other Methods of managing Addition, whereby the Number of Tens will be mark'd out in the Operation, which shall be presently explained; but I shall first make some larger Examples, and shew the Application of the preceding Rule to them.

Ex.

Ex. 3d.	Ex. 4th.	Ex. 5th.
868	87869	34
753	26100	596
209	7895	789
546	3746	6897
370	65432	2540
921	789576	4698
689	456789	246
796	324160	86532
<u>5152</u>	<u>1761567</u>	<u>78063</u>
		<u>65</u>
		180460

The Operation of Ex. 3. is thus, $6+9=15$; $+1=16$; $+6=22$; $+9=31$; $+3=34$; $+8=42$; which is 2 for the first place and 4 to carry to the 2d: thus, $4+9=13$; $+8=21$; $+2=23$; $+7=30$; $+4=34$; $+5=39$; $+6=45$; which is 5 for the 2d place, and 4 to carry to the 3d: thus, $4+7=11$; $+6=17$; $+9=26$; $+3=29$; $+5=34$; $+2=36$; $+7=43$; $+8=51$; which being the Sum of the last Column, is all written down; so the total Sum is 5152. — Observe that you are to read the Operation thus, $6+9=15$, then $15+1=16$, &c. But to prevent writing the last Sum twice, I have separated it from the Number to be next added to it by a Semicolon, by which you must understand, that it is added to the next Number.

[But not that these Expressions are all equal, viz. that $6+9$, $15+1$, &c. are equal.]

After the same manner examine the other two Examples, for your own Practice.

2. If there are many Numbers to be added, so that the Sum of every Column be a great Number, we may save the Memory being too much burdened, and the Work render'd thereby more difficult and uncertain, by various means; as,

First, By making a Point at every 40 or 50, or 100. Thus, Add up the Column till you have a Sum containing any Number of Tens you please, as 50 or 100, making a Point at the last Figure which makes up your Number of Tens, and if there is any excess over Tens, add it to the next Figure, and so go on thro' all the Column: When you have finished it, if the particular Sum after adding the last Figure is less than 10, set it down in the place of the total Sum (or Answer,) and for every Point carry as many Units to the next place as there are Tens in the Number pointed, (i. e. 3, 4, 5, or 10, for 30, 40, 50, or 100,) and if the last particular Sum exceed a Number of Tens, set down the Excess, and join that Number of Tens to the Number arising from the Points, and carry the Sum to the next Column; and thus go thro' them all.

So in the annex'd Example, which I point at 50, I proceed thus; $9+7$ is 16, and 8 is 24, and 7 is 31, and 9 is 40, and 8 is 48, and 9 is 57; here I point at the 9, and take the 7 which is over 50, and say, $7+6$ is 13, and 9 is 22, and 4 is 26, and 5 is 31, and 6 is 37, and 7 is 44, and 5 is 49, and 7 is 56, which makes a Point at the 7, and 6 over; with which I proceed, saying, $6+9$ is 15, and 8 is 23; therefore I write down 3 in the Sum, and I have 2 (for the 20) to be added to the Number of the two Points, viz. to 10 (for every Point is here 5) and so I have 12 to carry in to the next Column; which I add in the same manner.

To prevent blotting of Accounts, which is the Objection made against this Practice, you may make the Points upon a Shred of Paper applied to the Column.

There is also another useful Method of summing up long Columns, viz. by distributing the Numbers into Parcels of 9 or 10 Numbers in each, (so that the Sum of each Column in every Parcel shall never exceed 100) adding each Part by itself, and then adding their Sums into one total Sum. As in the following Example, which needs no further Explanation.

528	
269	
587	
675	
367	
986	
895	
764	
489	
76	
29	
38	
19	
27	
38	
27	
9	
<u>5803</u>	

E

D E.

DEMONSTRATION of the Rule of CASE 2.

678
 957
 834
 928
 653
 356
 890
 348
 269
 --- 5913
 372
 908
 735
 687
 943
 259
 167
 896
 725
 --- 5702
 Total 11615

This we deduce from the Nature of Notation, and the common Axiom, *That the Whole is equal to all its Parts*: Thus; every Number consists of as many Parts as there are significant Figures in it; and the Figures standing in the same Places of different Numbers are the similar or like Parts of these Numbers; which being therefore added together according to their simple Values, the Sum has all the same secondary or local Value, viz. That of the Place in which these Figures do all stand: But (by Cor. 4. Chap. 2.) any Number is equal to as many Tens as the Figures above the place of Units taken by themselves do express, and to that Number of Units more; (so 248 is 24 Tens + 8.) And again, 1, or any Digit standing in any place is equal to 10 times the Value of the same Digit in the preceding place: wherefore the Sum of the Digits standing all in the like places of different Numbers, is equal to as many Units of the Value of the next higher place as there are Tens in the Sum, and to as many Units of the place added as are over that Number of Tens. Hence we see the Reason of all the Parts of the Rule: for the Parts of Numbers added must be similar, else the Sum is false; so $4 + 7 = 11$, which supposes them to be both of one local Value; but if the 4 be in the place above the 7, the Sum is not 11, but 47: This explains the first Part of the Rule. Then, by carrying forward the Number of Tens found in the Sum of every Column (added according to

the simple Values) and adding that Number to the next Column (with which it has a similar Value) we actually add all the Parts of each of the Numbers given; and do add them so, as that all the Units of each particular Digit set down in the total Sum have one local Value, which is the same that the Parts to whose Sum it belongs have in the Numbers added; but the Parts are equal to the Whole, therefore the Numbers set down according to the Rule, express the true Sum.

As to the second Part of the Rule, viz. The carrying forwards the Number of Tens, we may see the Reason of it another way; Thus, suppose every Column added by itself, and the Sums set down separately so as the first Figure of the Sum stand in the place of that Column; and let all these Sums be again added in the same manner; and so on till we have the most simple Expression of the Total of the given Numbers. Now this is a very natural way of Operating, which carries its Reason with it; but it is also plain, that the carrying forward the Tens has the very same Effect; for it is only doing that all at once, which is here done at several steps, and is thereby a more compendious Way of finding the Number sought, and therefore preferable.

Example I.

	348
	695
	476
	587
	290

Sum of Units	26
Sum of Tens	370
Sum of Hundreds	2000
Total Sum	2396

Example

Example II.

5789	
6958	
4746	
8984	
4897	
6518	
7968	
9376	
<hr/>	
56	Sum of Units.
480	— of Tens.
5700	— of Hundreds.
49000	— of Thousands.
<hr/>	
06	Sum of Units.
130	— of Tens.
1100	— of Hundreds.
14000	— of Thousands.
40000	— of Ten Thousands.
<hr/>	
55236	Total.

§. 2. *Of the Proof of Operations in Arithmetick, and particularly of ADDITION.*

THE Proof of any Operation in Arithmetick is some other Operation, by which we are made more certain that the first Work is right performed, and the true Answer found.

It is different from the Demonstration of the Rule; for this shews that if the Rule is followed, the true Answer will be found: But the Proof supposes the Truth of the Rule, and only shews whether we have observed it in doing the Work.

The Proof of any Operation ought naturally to be another easier than itself, in which there is less hazard of erring; otherwise we are not more certain of the one than of the other: And if we be very scrupulous, we may require again a Proof of this second Work, by a third easier than the last; and so on till we have the most simple Operation. But a little practice puts us beyond the need of all this; and we may be satisfied with any one Proof, even tho' it is not an easier Work than that to be proved, because there is less hazard of Erring in both than in one. However, the easiest Proofs, if they are not tedious, are always preferable.

P. R. O. O F of A D D I T I O N.

If we should propose a more simple Kind of Operation for *Proof of Addition*, we have none but the adding Unit by Unit, already spoken of in *Notation*. But the simplicity of this Method is infinitely overbalanced by its tediousness; therefore we must be content with another Application of the same Rule; which may be made different ways.

1. If the Numbers have been added all together without distributing them into Parcels, (as has been explained) then by making such a Distribution, and adding them that way, we shall prove the Answer which was found the other way; for the total Sum must be the same in both Methods: But if it has been at first wrought this way, we may make a Proof by making a different Distribution.

2. Whatever way the Work is done at first, we may do it again the same way, only beginning at the upper Line, and adding downwards.

3. There is another way ingenious enough, and very easy; but as it supposes a small Capacity in Subtraction, it may seem not agreeable to good Order to propose it in this place: For tho' one Operation or Rule may no doubt be made serviceable to another, yet every thing ought to be in its Place; therefore you'll say, that according to the utmost strictness of Method, what requires Subtraction, ought to come after Subtraction. But the thing supposed is no more than that we can readily know the difference betwixt 9 and any Number less than 18; and this Capacity cannot but be already acquired by the practice of Addition, (of which Subtraction is but the Reverse.) There is also something in it which produces the same Effect as *Division*; but as it does not require the Rule of Division, that furnishes no Objection: Therefore I shall explain this Method here as its proper place.

Rule. Take each of the given Numbers separately, and add all their Figures together as simple Units; and in doing so, when you have made a Sum equal to 9, or greater than 9, but less than 18, neglect the 9, taking what's over and add to the next Figure; and go on so till you have gone thro' them all, and mark what is over or under 9 at the last Figure; but if the Sum of all the Figures is less than 9, mark that Sum. Do the same with each of the given Numbers, setting all these Excesses of 9 together in a Line, (in any Order,) then sum them up the same way, marking the excess of 9 as before, (or what the Sum is less than 9.) Lastly, do the same with the total Sum, and what is under 9, or over any Number of 9's in this, must be equal to the Excess (or Number less than 9) last marked; else the Work has not been right performed.

I shall explain this Practice by one Example.

Example.

$\begin{array}{r} 2743 \\ 4678 \\ 5265 \\ \hline 12786 \end{array}$	$\left. \begin{array}{l} 7 \\ 7 \\ 1 \\ 6 \end{array} \right\} \text{excess of 9}$	<p>Beginning with the upper Line, I work thus; $2 + 7 = 9$; then $4 + 3 = 7$, which I have set down on the Right. Then to the second Line, $4 + 6 = 10$, which is 1 over 9; then $1 + 7 + 8 = 16$, which is 7 over 9; which I set under the Figure last found. Again, to the third Line, $5 + 3 + 6 = 14$, which is 5 over 9; then $5 + 5 = 10$, which is 1 over 9. Then I do the same with the Line of the Figures now found, viz. $1 + 7 + 7 = 15$, which is 6 over 9; and finding the same Excess of 9's in the Sum (12786,) I conclude the Work is right performed.</p>
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Observe, It will have the same Effect if, instead of setting down the Excess of 9's in the several given Numbers, we carry the Excess of one Line into another, and only mark the last Excess; which ought to be the Excess in the total Sum.

Demonstration. In order to shew the Truth of this Rule, I must premise and demonstrate this *Lemma*, which will be useful also afterwards.

L E M M A.

The Figure that stands in any place of a Number, taken in its simple Value, is equal to what will remain after 9 is taken out of the compleat Value as oft as possible; i. e. after all the 9's contained in it are taken away.

For Example, If all the 9's contained in 700 are taken away, there remains the simple Number 7.

Demon. Any Figure standing in any place of a Number is equal to ten-times the Value of the same Figure in the next lower place, (by what has been shewn in Notation;) i. e. equal to 9 times + 1 time that Value, (because $9 + 1 = 10$.) But 9 times any Number is a precise Number of 9's; which being taken away, there remains once the Value of it in that next place, and this again is equal to 9 times + 1 time the Value of the same Figure in the next lower place, and the 9 times being taken away, the 1 time remains; and so on till you bring it down to the place of Tens, where it is equal to 9 times its simple Value + once that Value; and the 9 times taken away, there remains the simple Value: But

But thus we have supposed all the 9's to be taken out of it, and consequently the *Lemma* is true.

Corollary. The Sum of all the Figures in any Number, taken as simple Units, is the Remainder after as many 9's are taken out of that Number, as are to be found separately in the compleat Value of each of the said Figures (because each of these Figures taken simply, is the Excess of the 9's contained in that Part,) and if that Sum is less than 9, it is the Remainder after the 9's contained in the given Number are taken away: but if it is not less than 9, the Remainder, after all the 9's are taken out of it, is the Remainder of 9's in the given Number: For it is plain that there can be no more 9's in any Number than what are in the several Parts and in the Sum of the Excess of 9 in the same Parts.

Now from this *Lemma* and *Corollary*, the Demonstration proposed will be plain: For by adding the Figures of any Number according to the Rule, it is evident we find the Excess over all the 9's contained in their Sum (taken as simple Units.) And this is the Excess of all the 9's contained in the said Number by the *Corollary*. But again, the Excess of 9's in each of two or more Numbers being taken separately, and the Excess of 9's taken also out of the Sum of the former Excesses, it is plain this last Excess must be equal to the Excess of all the 9's contained in the Total of all these Numbers, (the Parts being equal to the Whole.)

SCHOLIUMS.

1. In the Demonstration of the preceding *Lemma*, I have taken it for a Truth, that 9 times any Number is a precise Number of 9's; *i. e.* that it is equal to that Number of times 9, (without any thing over.) For *Example*, that 9 times 7 is 7 times 9; for the Demonstration of which (if it is required) I refer you to Chap. 5. as I have done already in a like Case.

2. To this Proof it is commonly objected, That a wrong Operation may appear to be true; which must be owned: for if we change the Places of any two significant Figures in the Sum, it will still appear right. So in the preceding Example, the true Sum is 12786; But suppose, thro' mistake, it had been 12768; it is plain this Method of Proof would make it appear right, because there is the same Excess of 9's where there are the same Figures, whatever order they stand in: But then consider, a true Sum will always appear true by this Proof, (for that is demonstrated) and to make a false Sum appear true, there must be at least two Errors, and these opposite to one another; *i. e.* one Figure greater than it ought to be, and another as much less; and if there are more than two Errors, they must always balance among themselves; *i. e.* the Sum of the Figures that are greater than they ought to be, must always be equal to the Sum of the Figures that are deficient; else it is plain, a false Sum will not appear to be right. But now if we consider what an exceeding great Chance there is against this particular Circumstance of the Errors, and how simple the Proof-work itself is, we may trust to this Proof as safely as to any other.

C H A P. IV.

SUBTRACTION of *Abstract Whole Numbers.*

D E F I N I T I O N.

SUBTRACTION is the taking one Number out of another; or finding the Difference betwixt two Numbers; *i. e.* the most simple Expression of that Number whereby the greater of two given Numbers exceeds; or the lesser comes short of the other. *Example.*

Example. The Difference betwixt 8 and 3 is 5: Betwixt 48 and 19 is 29.

Observe also, That for distinction the greater Number is called the *Subtrahend*, and the lesser the *Subtractor*; and the Number sought, or the Difference, is called also the *Remainder*, *i. e.* what remains after the lesser is taken out of the greater.

SCHOLIUMS.

1. As the Effect of *Subtraction* is plainly the Reverse of *Addition*; so is its Operation: Wherefore we have this first to observe, That by a continual retracting Unity after Unity (by the reverse of what was done in Addition) the difference of any two Numbers may be found; which shews an immediate Connection betwixt this Work and the Rule of Notation. But the insufferable tediousness of this Method is removed by the following Rule, whereby that is done by a few easy steps, which cannot be done all at once, and ought not to be done by more steps than are necessary: and this we owe to the Method of Notation. But we must also observe, That as the subtracting Unity from any Number is the most simple Case, so it is immediately contained in the Rule of Notation, and presupposed in the following *Problem*; and is also the only Method by which one Digit can be subtracted from another; which therefore I make the first *Case* of the following *Problem*, as that upon which all other *Cases* depend.

2. This Sign or Character — set betwixt two Numbers, signifies the *Subtraction* of the one from the other; and is a complex or indefinite way of representing the *Difference*. Thus $7 - 4$ signifies that 4 is taken from 7; and we read the Sign by the word *less*. *Examp.* $7 - 4$ is read, 7 less 4: and thus it expresses the difference in a complex Manner, by the Whole and Part taken away. And if more Numbers are successively subtracted, we prefix the same Sign to each of them; thus, $12 - 4 - 3$ signifies that 4 is taken from 12, and then 3 from the first Remainder; or, which is the same thing, that 4 and 3 both (*i. e.* 7) is taken from 12. By this Sign we explain in a neat and brief way the Work of particular Examples in *Subtraction*; as you'll presently see.

But the principal Use of this Sign of *Subtraction* is for the Expression of the difference of Numbers represented by Letters in the *Algebraick Art*. Thus, $A - B$ expresses the difference of A and B; $A - B - C$, the difference of A and $B + C$; for by taking all the Parts of one Number successively from another, we take the Whole of that Number from this. And this is the *General Rule of the Literal Subtraction*: other particular Cases you'll earn in another place.

PROBLEM.

To Subtract one Number from another; or, to find their Difference.

CASE 1. To find the Difference betwixt two Digits.

Rule. This is to be done by a continual retracting each Unit of the *Subtractor* successively from the *Subtrahend*, and expressing the Differences gradually, according to the *Rule of Notation*.

Example. $9 - 4 = 5$. For $9 - 1 - 1 - 1 - 1 = 5$. The gradual Series of Differences being 8, 7, 6, 5; because $9 - 1 = 8$, $- 1 = 7$, $- 1 = 6$, $- 1 = 5$; the Number sought.

SCHOLIUM. As all other Cases necessarily suppose this one; the Answers of all its Examples ought to be ready in the Memory: and therefore I shall put them all in the following Table, for their Use who are mere Novices in *Arithmetick*. But I must here observe, That as this is only the Reverse of *Addition*, whoever is Master of that, (as they ought to be before they enter on *Subtraction*) will be able at once to pronounce the difference of any two Digits.

T A B L E for finding the Differences of any two Digits.

1	2	3	4	5	6	7	8	9	0
0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9

The Use of this Table will, I think, be obvious: Seek the *Subtractor* on the Right-side Column, and against it under the *Subtrahend*, found in the Upper-Line, is the difference sought. *Examp.* Under 7 on the Head, and against 4 on the Side, you find 3, the difference of 7 and 4. Or also thus, Take the *Subtrahend* on the Head, and in the Column under it, seek the *Subtractor*, against which on the Side is the Difference.

CASE 2. To Subtract any Number from another.

Rule 1. Write the one under the other according to the Order of Places, (and most commonly the *Subtrahend* is set above the other, tho' this is not necessary.) Then, *Secondly*, take the difference betwixt each Figure of the *Subtractor* and its Correspondent in the like place of the *Subtrahend*, beginning at the place of Units, and set the Remainders under them in order; but if the Figure in any place of the *Subtrahend* is less than its Correspondent in the *Subtractor*, add 10 to that Figure, and subtract from the Sum, and set down the Digit which remains; then add 1 to the next Figure of the *Subtractor*, and take the Sum from its Correspondent in the *Subtrahend*; and go on so, adding in the same manner whenever the Figure of the *Subtrahend* is less. All the Figures written down, express the difference sought.

Subtrahend 876 *Examp. 1.* The difference of 876 and 524 is 352, as in the Margin. The Operation is thus, $6 - 4 = 2$, which I set down; then $7 - 2 = 5$, and $8 - 5 = 3$; and the difference sought is 352.

Subtr. 74625 *Examp. 2.* The difference of 74625 and 47382 is 27243. Whose Operation is thus, $5 - 2 = 3$; then $2 - 8$ is impossible, therefore I take 12, and say, $12 - 8 = 4$; then (because 10 was added to the last place of the *Subtrahend*) I add 1 now to the *Subtractor*, and say, $1 + 3 = 4$; then $6 - 4 = 2$. Again, $4 - 7$ cannot be, therefore I take 14, and say, $14 - 7 = 7$. Again $1 + 4 = 5$, and $7 - 5 = 2$. And so the complete difference is 27243.

You may examine these few more Examples in the same manner.

Examp. 3.

87234
53890
33344

Examp. 4.

2835
925
1910

Examp. 5.

345
79
266

Examp. 6.

58034
928
57106

SCHOLIUM. This Rule supposes we can readily in our Mind subtract any Digit from the Sum of any lesser Digit and 10; which may be easily admitted: but to make the Difficulty less, (if there can be any) we may do the Work thus; Subtract from the 10, and add the Remainder to the Figure of the *Subtrahend* to which the 10 should have been added, and set this Sum down. So in *Examp.* 5. say 9 from 5 cannot, but from 10, and 1 remains; then $1 + 5 = 6$; which is set down.

DEMONSTRATION of the Second Case.

1. Where all the Figures of the *Subtractor* are less than their Correspondents in the *Subtrahend*, the difference of the Figures in the several like places set in the same place, must all together make the true difference sought; because as the Parts make up the Whole, so must the differences of all the similar Parts of any two Numbers make the total Difference of the Wholes, of which these are the similar Parts.

2. Where any of the Figures of the *Subtrahend* is less than the Correspondent in the *Subtractor*, the 10 which is added by the Rule you are to suppose to be the Value of an Unit taken from the next higher place, (which by *Notation* is equal to 10 in this Place,) and then the 1 added to the next place of the *Subtractor* is to diminish the next place of the *Subtrahend* by 1 more than is contained in the *Subtractor*, because that 1 was supposed to be already borrowed from it, and applied to the preceding Place according to its Value there: so that instead of adding 1 to the *Subtractor*, we may take 1 from the *Subtrahend*; i. e. take the *Subtractor* from the *Subtrahend* Figure lessened by 1. But the Effect is the same, since either way that 1 is taken from the *Subtrahend*; as it ought to be, since it is already applied to the last place; which is only taking from one Part, and adding as much to another, whereby the Total is never changed: And by this means the *Subtrahend* is resolved into such Parts as are each greater than (or equal to) the similar Parts of the *Subtractor*. So in order to subtract 26 from 52, resolve the 52 into these Parts, 40 and 12, so that the 12 correspond to the 6 of 26 the *Subtractor*, and the 40 to the 20; which is in effect done by the Rule: for we say, 6 from 2 cannot, but from 12, and 6 remains; then it is plain, that for 5 we have but 4 in the next place, and it is the same thing to say 2 from 4, or 3 from 5.

But there is another thing to be accounted for, which the Rule supposes, viz. That the Difference betwixt any Digit and the Sum of any lesser Digit and 10, will always be less than 10; the Truth of which is plain; for since that lesser Digit wants at least 1 of the greater, (to make a Part equal to the *Subtractor*) that 1 being taken from the 10 added, there cannot remain above 9.

SCHOLIUM. If it be proposed to subtract two or more Numbers from any one, or one from more; or lastly, more than one from more than one; the best and most simple way is, first to add these more together, and then subtract: So let it be proposed to subtract 560 from $467 + 235$, as in Example 1. below; or, $345 + 432$ from 978, as in Example 2. or, $3072 + 5678$ from $2578 + 9631$, as in Example 3. Also when more Numbers are to be subtracted out of one, it may be done by taking away first one of them, and then out of the Remainder take another, and so on till they are all subtracted; and this is called *Continual Subtraction*.

Example 1.

	467
	<u>235</u>
Sub ^d .	702
Sub ^r .	<u>560</u>
Diff.	<u>142</u>

Example 2.

Sub ^d .	978
	<u>345</u>
	432
Sub ^r .	<u>777</u>
Diff.	<u>201</u>

Example 3.

	2578
	<u>9631</u>
Sub ^d .	12209
	<u>3072</u>
	5678
Sub ^r .	<u>8750</u>
Diff.	<u>3459</u>

The PROOF of Subtraction may be made either by Addition or Subtraction.

1. By *Addition*, thus; Let the Remainder be added to the *Subtractor*, and the Sum ought to be equal to the *Subtrahend*: For this is restoring back what was before taken away.

2. By *Subtraction*, thus; Subtract the Remainder from the *Subtrahend*; and this Remainder ought to be equal to the former *Subtractor*: Because which soever of the two Parts that make up any Whole is taken away, the other remains. I leave you to apply these Proofs in the preceding Examples.

If we have subtracted more Numbers out of one by continual Subtraction, the Proof of this Work will plainly be, *Adding all the Subtractors and the last Remainder into one Sum, which must be equal to the Subtrahend.*

CHAP. V.

MULTIPLICATION of *Whole and Abstract Numbers.*

DEFINITIONS.

1. **M**ULTIPLICATION is the taking any Number a certain number of times: or, finding a Number which shall contain any given Number a certain proposed number of times; i. e. as oft as that other Number contains Unity. For Ex. If it is proposed to take 48, 7 times; or find that Number which contains 48, 7 times; the Answer by the following Rule, is, 336.

2. The Number to be multiplied is called the **MULTIPPLICAND**; the Number by which it is multiplied, (or the number of times it is to be taken) is called the **MULTIPLIER**; and the Number found is called the **PRODUCT**. The *Multiplicand* and *Multiplier* are also called the **FACTORS** (of the *Multiplication*) without distinction; because they make the *Product*, or Number sought.

SCHOLIUMS.

1. I know there may be made a more general Definition of *Multiplication*, comprehending in it also what is called, *Multiplication by Fractions*: But that being really a mixt Operation of *Multiplication* and *Division*, I thought it more reasonable to make the Definition here to agree only to whole Numbers, which is proper and pure *Multiplication*, according to the more common Sense of the Word; and when you learn the other, it is only joining them both together to make that more general Definition: which I shall do in its proper place.

F

2. The

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2. The Sense and Effect of *Multiplication* (of whole Numbers) is the same with *Addition*; for it's plain, that if we take the *Multiplicand*, and write it down as oft as there are Units in the *Multiplier*, the Sum of all these, taken by *Addition*, is the Number sought.

For *Ex.* to multiply 48 by 7, or to take 48, 7 times; I set down 48, 7 times (as in the Margin), and find the Sum 336. But *Multiplication* is a Method of finding the same Number more easily and expeditiously. For *Ex.* to multiply any Number by 468; what a tedious and intolerable thing would it be, to write down the *Multiplicand* 468 times? But by the following Rules this is prevented, and the Number sought is found by an easy Operation.—*Multiplication* then is only a compendious *Addition*, limited to that particular Case wherein all the Numbers to be added are equal to one another, (or the same Number:) For it's this Circumstance that affords us a more easy Method of working than by the general Rule of *Addition*. Yet there are some simple Cases which admit of no Compend: These are the *Multiplication* of Numbers under 10, or the Digits, by one another; which are the *primitive Operations* in *Multiplication*, upon which all other Cases depend. We must therefore explain *Multiplication* also in two Cases, as we have done *Addition*. But whereas the *primitive Cases* of *Addition* depend immediately upon the Rule of *Notation*; the *primitive Cases* of *Multiplication* depend immediately on *Addition*. For we need not go back to *Notation* for these; since we have, by an intermediate Step in *Addition*, gained an easier way of doing them.

3. This Sign or Character \times set betwixt two Numbers, signifies the *Multiplication* of the one by the other (taking either of them for the *Multiplier* or *Multiplicand*; which does not alter the Product, as will be afterwards demonstrated) and is a Complex or indefinite way of representing the Product. Thus 7×3 signifies that 7 is multiplied by 3, or 3 by 7; which we read 7 times 3, or 3 times 7; whereby the Product is expressed in a complex manner by the *Factors*: And if more Numbers are successively or continually (as it's called) multiplied together, the same Sign is prefix'd to each successive Factor. Thus $4 \times 6 \times 3$ expresses the Product made by multiplying 4 by 6, and this Product again by 3; and so of more Factors, (which will still be the same Product in whatever order the Factors are applied, as will be demonstrated.) By this Sign we explain in a neat and brief way the *Multiplication* of particular Examples, as you'll presently see.

But the principal Use of this Sign of *Multiplication* is in the Algebraic Art, to express the Product of two or more Numbers in Letters. Thus, $A \times B$ is the Product of A and B; also $A \times B \times C$ the Product of A, B and C, and so on. But when two or more Numbers are expressed, each by one Letter, the Product is also expressed by these Letters set together without the Sign. Thus AB is the Product of A and B, ABC that of A, B, and C. This is the general Rule of the *Literal Multiplication*; other particular Cases you'll learn afterwards.

P R O B L E M.

To multiply any Number by another.

CASE I. *To multiply one Digit by another.*

Rule. Add the *Multiplicand* to itself as oft as there are Units in the *Multiplier*, and you have the Product sought.—*Examp.* To multiply 6 by 4, I say $6 + 6 (=12) + 6 (=18) + 6 = 24$, the Number sought. But the Products of all the Digits ought to be ready in the Memory; which are easily got by the help of the following Table.

TABLE

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TABLE of *MULTIPLICATION* for the simple Digits, or Numbers less than 10.

1	2	3	4	5	6	7	8	9	1
	4	6	8	10	12	14	16	18	2
		9	12	15	18	21	24	27	3
			16	20	24	28	32	36	4
				25	30	35	40	45	5
					36	42	48	54	6
						49	56	63	7
							64	72	8
								81	9

USE of the TABLE.—Seek the greater of the two Digits in the upper Line, and under it against the lesser, taken in the right-side Column, is the Product sought. *Ex.* To multiply 8 by 4, I take 8 in the upper Line, and under it against 4 I find 32 the Product. Again, to multiply 7 by 9, take 9 in the upper Line, and under it against 7 on the side, you have 63 the Product sought.

DEMON. The Construction of the Table is plainly this: The 9 Digits being set down in the upper Line, each of them is considered as a Multiplicand; and is added to itself successively, as oft as there are Units in every Digit not exceeding itself; and the Multipliers are set in a Column on the side against the respective Sums. For *Example*, the Multiplications of 6 from the upper Line are carried no farther than 6 on the side; and so of the rest. This explains the Reason of the Rule when the greater Number is proposed as the Multiplicand; or where the Multiplicand and Multiplier are equal: But, where the lesser is proposed as the Multiplicand, yet we apply the greater to the Table, as if it was the Multiplicand, [which I shall here suppose to be the same in effect, and afterwards it shall be demonstrated, *viz.* that 6 times 8 is the same as 8 times 6, and so of any other two Factors.]

CASE II. To multiply any two Numbers into one another.

Rule 1. Make any of the two Factors the *Multiplicand*, or *Multiplier*; but it will be generally most convenient to make that the *Multiplier* which has fewest significant Figures: Then, tho' there's no matter in what order the Factors are set down, yet, when it can be done, 'tis convenient to write the Multiplier under the Multiplicand; so that the first significant Figure on the Right-hand of this be over the first significant Figure of the other, [whether these Figures be in the place of Units or not. See the Examples.] Then, in case there are 0's standing on the Right-hand of either or both Factors, neglect them as if they were not there, and proceed thus:

2. Begin with the first significant Figure of the Multiplier, and by it multiply every Figure of the Multiplicand, one after another, (by *Case I.*) beginning at the first significant Figure, and proceeding in order to the last Figure on the left; and write down the Products. *Thus*, If the first Product (or, that of the first significant Figure of the Multiplicand) is less than 10, write it down; but if it exceeds 10, write down what's over any Number of 10's, and carry that Number (*i. e.* 1 for every 10) to the Product of the next Figure. If this Sum is less than 10, write it down; but if it exceeds 10, write down the Excess of 10's, and carry the Number of 10's to the Product of the next Figure, and so on: setting the Figures to be written down all in a Line after one another, orderly, from the Right to the Left. Having thus gone thro' all the Figures of the *Multiplicand*, (not omitting the 0's that stand mixt with other Figures) write down the complete Product of the last Figure with the 10's of the preceding added to it.

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3. Make the same Operation upon the *Multiplicand*, with every significant Figure of the *Multiplier*; (passing all the 0's) taking them in order as they stand towards the Left; and setting the several Products under one another: Thus, Set the first Figure of the Product made by every Figure of the *Multiplier*, at the same distance from the first Figure of the preceding Line, as their respective Figures in the *Multiplier* are; and place the rest of the Figures in order towards the Left, under the preceding.

4. Add all these particular Products into one Sum; taking them as they are set in the Columns under one another: And if the first significant Figure of both *Factors* was in the place of Units, this Sum is the Product sought (Ex. 1, 2, 3, 4, 5.) otherwise set as many 0's before it as stand before the first significant Figure of both *Factors*, (as in Ex. 6, 7, 8, 9.)

I shall next apply this Rule in *Examples*.

Ex. 1. To multiply 642 by 4; I say $4 \times 2 = 8$, which is set down; then
 642 Multiplicand. $4 \times 4 = 16$, for which I write down 6 and carry 1; then 4×6
 4 Multiplier. $(= 24) + 1 = 25$, which being written down, the Product is 2568.
2568 Product.

Ex. 2. To multiply 85065 by 8, I work thus, $8 \times 5 = 40$, for which I write
 85065 M^d. down 0, and carry 4; then $8 \times 6 (= 48) + 4 = 52$, for which I write
 8 M^r. down 2 and carry 5; then $8 \times 0 (= 0) + 5 = 5$, which I write down;
680520 Pr. then $8 \times 5 = 40$, for which I write down 0, and carry 4; then, 8×8
 $(= 64) + 4 = 68$; which being written down, the Product is 680520.

Ex. 3. To multiply 84653 by 469, I work thus; beginning with 9
 84653 M^d. (in the Units place of the Multiplier) I multiply by it the whole
 469 M^r. Multiplicand, in the manner of the preceding *Examples*. Then
 761877 Prod. by 9. I take the next Figure of the Multiplier, 6, and by it in the same
 507918 ——— by 6. manner multiply the whole Multiplicand; setting the first Figure
 338612 ——— by 4. of this Product, 8, under the second Figure of the preceding, and
 39702257 Total Prod. the rest in order; making the same Operation with the next (and
 of this Product (*viz.* 2) under the second of the preceding, and the rest in order. All
 these Products summed up, as you see in the Ex. gives the total Product.

Ex. 4. To multiply 6452 by 806: After multiplying by 6, I pass to 8; and because
 6452 there is one 0 betwixt them, the first Figure of the last Line is set under the
 806 third place of the former, *i. e.* with one place betwixt the first Figures of the
 38712 two Lines, because of the one 0 betwixt the Multipliers.
 51616
5200312

Ex. 5. To multiply 46007859 by 380046: After making the Product of
 46007859 6 and 4, I pass to 8; setting its first Product two places distant from
 380046 that of the preceding Line, because of the two 0's betwixt 4 and 8.
 276047154
 184031436
 368062872
 138023577
17485102781514

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In the four following Examples, wherein the first significant Figure of each *Factor* is not in the place of *Units*, the Application of the Rule is so plain, that I need make no more words about it.

Examp. 6.	Examp. 7.	Examp. 8.	Examp. 9.
$\begin{array}{r} 467 \\ 2800 \\ \hline 3736 \\ 934 \\ \hline 1307600 \end{array}$	$\begin{array}{r} 584000 \\ 93 \\ \hline 1752 \\ 5256 \\ \hline 54312000 \end{array}$	$\begin{array}{r} 46000 \\ 2700 \\ \hline 322 \\ 92 \\ \hline 124200000 \end{array}$	$\begin{array}{r} 376890 \\ 5004000 \\ \hline 150756 \\ 188445 \\ \hline 188595756000 \end{array}$

In order to the Demonstration of the preceding Rule, and for the sake of some other special Rules following, we must premise these Truths, as belonging to the Theory of *Multiplication*.

L E M M A I.

If one Number is multiplied by another, the Product will be the same as if this other be multiplied by the former; i. e. Any one of the two Factors may be made Multiplier or Multiplicand, the Product will be the same.

Examp. 4 times 7 = 7 times 4. A times B = B times A.

Demon. A small Attention to the Idea of Numbers will make this Truth evident; and therefore few Writers think it needs any Demonstration. However, as it is capable of one, and some may require it, I shall satisfy them. Thus;

Any Number B is only a certain Collection of *Units*; wherefore A times B is equal to A times each *Unit* in B, taken separately and added together: but A times 1 is the same thing as A, or 1 time A, (from the Definition of Number:) Therefore A times each *Unit* in B is equal to A, (or 1 time A) taken as oft as there are *Units* in B; i. e. B times A. Therefore A times B = B times A.

Or take this other more sensible Demonstration.

Suppose any two Numbers, A, B; let the *Units* of A be represented by a Row of Points, as in the Margin: Repeat this Row as oft as there are *Units* in B, and set them orderly under one another; then it is plain, that there will be as many Columns of Points as there are Points or *Units* in A; each of which Columns has as many Points as there are *Units* in B. Therefore the whole Number of Points which were at first made equal to B times A, (by repeating the Row A, B times) becomes necessarily equal to A times B.

A

 B &c.

 &c.

L E M M A II.

If three or more Numbers are proposed to be continually multiplied, the last Product will still be the same, in whatever Order the Factors are taken.

Examp. $3 \times 5 \times 7 = 3 \times 7 \times 5 = 7 \times 3 \times 5 = 7 \times 5 \times 3 = 5 \times 3 \times 7 = 5 \times 7 \times 3$.

DEMON. CASE I. If there are 3 Factors A, B, C; then since the Product of 2 Factors is the same whichever of them is *Multiplier*; i. e. A times B = B times A: Therefore the thing to be proved here is only, That whichever of the 3 Factors is cast in the last place, the Product is still the same, viz. that $ABC = CBA = ACB$. For since $AB = BA$; therefore $ABC = BAC$. For the like Reason $CBA = BCA$, and $ACB = CAB$.

I

Now

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Now then, to prove that $ABC = CBA = ACB$: First, taking the *Factors* in the Order ABC , expresses the Product of A by B , or B times A , and this Product AB taken C times; which makes C times B times A , or A taken C times B times; *i. e.* CB times A : So that $AB \times C = A \times CB$, or $CB \times A$; *i. e.* $ABC = CBA$. Again, ACB signifies A taken C times, and this Product AC taken B times; which is B times C times A ; *i. e.* A taken B times C times; or, which is the same, C times B times: but this, *viz.* $A \times CB$ or $CB \times A$ was before shewn to be equal to $AB \times C$, therefore $ABC = CBA = ACB$.

CASE II. If there are more *Factors* than 3, then I prove the Truth proposed thus: I say, if it is true of any particular Number of *Factors* more than 2, it is therefore true if we take in one *Factor* more: but it is true of 3 *Factors*, as shewn above; therefore it is evidently true of 4, and hence again it is true of 5, and so on for ever. What is to be proved then, is the first Part, *viz.* the Connection of the Truth of the Rule for any Number of *Factors* more than 2, with the next Case, or one *Factor* more. Which I prove thus;

Let $ABCD, \&c.$ be a Product of any Number of *Factors*, in which it is supposed to be no matter in what Order they are taken; therefore I may cast any of its Terms last, the Product will still be equal, *viz.* $ABCD, \&c. = ABC, \&c. \times D = ABD, \&c. \times C = ACD, \&c. \times B = BCD, \&c. \times A$. Now, let another Term X be taken into the Question, then it is plain, that whatever *Factor* of the whole $A, B, C, D, X, \&c.$ we suppose to be last employed, the various Orders in which the preceding *Factors* may be employed, produce the same Effect, by supposition; wherefore all the various Orders of taking the whole *Factors*, wherein any particular *Factor* keeps the last place, produce the same Effect; because the various Orders of the preceding Terms have no different Effect by supposition. And therefore what remains to be proved is only this, *viz.* That the Products are all equal which are made by the several Orders wherein different *Factors* are put in the last place, which will easily appear thus: Since by supposition $ABCD, \&c. = ABC, \&c. \times D = ABD, \&c. \times C$, and so on, putting each *Factor* last: Therefore $ABCD, \&c. \times X = ABC, \&c. \times D \times X = ABD, \&c. \times C \times X$, and so on. But we may make X and the *Factor* preceding it, change places in each of these Expressions; the Product will still be the same by Case I. because it is a Product of 3 *Factors*, thus; $ABC, \&c. \times D \times X = ABC, \&c. \times X \times D$. Also, $ABD, \&c. \times C \times X = ABD, \&c. \times X \times C$, and so on; whereby each *Factor* is cast in the last place. Wherefore these Products are all equal, being each equal to the Product of another Order, and all these other Orders equal.

COROL. If two Numbers are proposed to be multiplied together, it is the same thing to do it by the *General Rule* at once; or, if one of the *Factors* is equal to the continual Product of two or more Numbers, then we may multiply the one given *Factor* first by one of these Numbers that produce the other, and then this Product by another of them, and so on thro' them all: Thus, $24 = 4 \times 6$. Therefore $62 \times 24 = 62 \times 4 \times 6$.

SCHOLIUM. What an *Aliquot Part* is, has been already explained. And from Lem. 1. it is evident, that if two Numbers are multiplied together, each of them is an *Aliquot Part* of the Product, and the other is the Denominator of the Part. So, because $4 \times 8 = 32$; therefore 8 is $\frac{1}{4}$ of 32, and 4 is $\frac{1}{8}$ of 32. Universally $A = \frac{1}{B}$ of AB , and $B = \frac{1}{A}$ of AB . Whence this follows, that if we multiply any Number by the *Aliquot Part* of another, and then this Product by the Denominator of that Part, the last Product is the same as if the first Number were multiplied by that other Number; which is a Truth manifest also from the nature of an *Aliquot Part*, without regard to this first Lemma. Again observe, That the Product of two Numbers is very naturally called, the *Multiple* of either of the *Factors*; and it is said to be a *Multiple* of it by the other *Factor*: So 48 is a *Multiple* of 8 by 6, or of 6 by 8. Also if two Numbers are multiplied by the same Number, the Products are called *Like-Multiples* of these Numbers, as 3×4 and 3×7 are *Like-Multiples* of 4 and 7. Now, as *Multiple* and *Aliquot Part* are directly opposite;

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so from the nature of them, this follows; That the Product of one Number by the *Aliquot Part*, or *Multiple* of another, is the like *Aliquot Part* or *Multiple* of the Product of these two Numbers. Thus; suppose $B = CD$; then is AC the $\frac{1}{D}$ Part of AB ($= ACD$), and AB ($= ACD$) is the *Multiple* of AC by D .

L E M M A III.

If either or both of two Numbers are any how distributed into two or more Parts; and if each Part of the one Number is multiplied into the other Number, or into each Part of the other; the Sum of all these Products is equal to the Product of the two given Numbers.

Examp. $3 + 4 = 7$, and $6 \times 7 = 3 \times 6 + 4 \times 6$. Again, $15 = 9 + 6$, also $8 = 3 + 5$. Therefore, $8 \times 15 = 3 \times 9 + 3 \times 6 + 5 \times 9 + 5 \times 6$. Universally, if $A = B + C$, then is $AD = BD + CD$; and if $D = E + F$, hence $AD = BE + BF + CE + CF$.

DEMON. The Reason is manifest; because the Whole being nothing else than the Sum of all the Parts, when one Number or every Part of it is multiplied into every Part of another Number, then is the one Whole multiplied into the other; so that the Sum of the Products made by the Parts must be equal to the Product of the Whole.

DEMONSTRATION of CASE II. of the preceding PROBLEM.

I. When the first significant Figure of each *Factor* is in the place of Units, (*Examp.* 1, 2, 3, 4, 5.) the Reasons of the Rule are these:

(1.) That either Number may be made *Multiplier* or *Multiplicand*, is demonstrated in *Lemma 1*. But making that one *Multiplier* which has fewest significant Figures, is most convenient, because it makes fewest partial Products.

(2.) If the *Multiplier* is a Digit, (*Ex.* 1, 2.) then by multiplying every Figure, *i. e.* every Part, of the *Multiplicand*, we multiply the Whole; and by writing down the Products that are less than 10, or the Excess of 10's, in the places of the Figures multiplied, and carrying the Number of 10's to the Product of the next place, we do hereby gather together the similar Parts of the respective Products; and so do the same thing in effect as if we wrote down the *Multiplicand* as oft as the *Multiplier* expresses, and added them up: For the Sum of every Column is the Product of the Figure in the place of that Column, whereby it appears that the Addition, and the Multiplication according to this Rule, have the same Effect: (see *Ex.* 1.) Wherefore the Rule is right.

(3.) When there are more significant Figures than one in the *Multiplier*, as in *Ex.* 3, 4, 5. then, by applying them separately, we resolve the *Multiplier* into Parts; and if the true Products made by each of these Parts multiplying the whole *Multiplicand*, are added together, the Sum is the Product sought, (by *Lem.* 3.) Again; the Product made by each Figure of the *Multiplier* taken in its simple Value, is truly found by the Rule; as shewn in the preceding Article; and by placing these Products with respect to one another, so as the first Figure of each Product stands under that Figure of the first Product, (or Product of the Figure in the Units place of the *Multiplier*), which is in the same place as the multiplying Figure stands in the whole *Multiplier*; these Products are added together according to the true Value they ought to have, by considering the multiplying Figure in its complete Value. For *Examp.* The Product by the second Figure, or Place of 10's, is set and added to the Product of the first Figure, as if a Cypher or 0 had been prefix'd to it, whereby it is made 10 times as much as were the multiplying Figure in the place of Units, as it ought to be, since that Figure is in the place of Tens, and not of Units; (see *Schol.* after *Lem.* 2.) The same Reason holds in all the other Figures of the *Multiplier*. Therefore the Sum of the Products taken according to the Rule, is the

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the true Product sought. And the Reason why we don't actually prefix these o's, is, because it were superfluous as to the Sum. See these Examples.

Ex. 1. $6784 \times 3 = 20352$. For $6784 = 6000 + 700 + 80 + 4$. And the Operation by Parts is thus:

$$\begin{array}{r|l} 4 \times 3 = 12 & \\ 80 \times 3 = 240 & \\ 700 \times 3 = 2100 & \\ 6000 \times 3 = 18000 & \\ \hline & 20352 \end{array} \quad \text{or thus, } \begin{array}{r} 6784 \\ 6784 \\ 6784 \\ \hline 20352 \end{array}$$

Example 2.

Common Way.

$$\begin{array}{r} 68749 \\ 853 \\ \hline 206247 \\ 343745 \\ 549992 \\ \hline 5864289 \end{array}$$

Or Thus.

$$\begin{array}{r} 68749 \\ 853 \\ \hline 206247 \text{ Prod. by 3.} \\ 3437450 \text{ Prod. by 50.} \\ 54999200 \text{ Prod. by 800.} \\ \hline 58642897 \text{ Total Prod.} \end{array}$$

II. When the first significant Figure of either or both *Factors* does not stand in the place of *Units*: Then,

(1.) When it is so only in one of the *Factors*, (as Ex. 6, 7.) then by taking that *Factor* as if the first significant Figure were in the place of *Units*; i. e. neglecting all the o's that stand on the right, we do indeed work only with the 10th or 100th, &c. part of it; therefore the total Product found is accordingly but the 10th or 100th Part of what it ought to be: and therefore to have the true Product, we ought to multiply the last Product by 10, 100, &c. (by *Schol. Lem. 2.*) which is done by setting as many o's before the Product as were before the Multiplier.

(2.) When it is so in both *Factors* (as in Ex. 8, 9.) there is the same Reason for setting before the Product the o's that belong to the one *Factor*, as those belonging to the other: For after correcting the Product by the o's of the one *Factor*, it wants to be corrected again by those of the other: Therefore when there are o's belonging to both, they ought all to be set before the Product.

Therefore this *Rule* is true in all possible Cases.

P R O O F of M U L T I P L I C A T I O N.

In the first place, we must observe, That the multiplying of one Digit by another has no other Proof than by Addition: But the Table being examin'd and found true, we are to depend upon that or Memory for these simple Operations; the Proof here designed being for other Cases where either or both *Factors* exceed 9: And this may be done several ways. Thus,

1. By *Multiplication* itself: For if we change the *Factors*, and make that the *Multiplier* which was before the *Multiplicand*, we ought to have the same Product, by *Lem. 1.*

2. By *Division*: But this cannot be applied till that *Rule* is learned.

But both these Methods are too tedious to be of Use; and therefore,

3. The most convenient and easy Proof is by help of the Number 9, like what we have in *Addition*; which is performed thus:

Cast

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Cast out the 9's out of the two *Factors* (in the manner taught in the like *Proof of Addition*) and mark the Number that is under or over 9's in each; if either of them is 0, pass immediately to the Product, and cast out the 9's of it; the Excess ought here to be 0, which makes the Proof. But if the Number under or over 9's is not 0 in either Factor, then multiply these two Numbers together, and mark also what is under or over 9's in their Product; this Number and what is over 9's in the total Product of the Example ought to be equal.

Ex. 1.

476
68
—
3808
2856
—
32368

In this first Example, the Excess of 9's in the *Multiplicand* is 8, in the *Multiplier* it is 5; then $8 \times 5 = 40$, and so the Excess of 9 is 4; exactly equal to the Excess of 9's in the Product 32368.

Ex. 2.

87
26
—
522
261
—
3132

In the second Example, the Excess of 9's in the *Multiplicand* is 0, therefore I pass immediately to the Product, where I find it is also 0.

Demonstration. The Reason of the Practice of casting out the 9's in the several Numbers has been already demonstrated in *Addition*; and the Reason of the rest of the Work will easily appear, thus: 1. If either *Factor* is a precise Number of 9's, (i. e. when there is no Excess) as in Example 2. it is plain the Product must be so too, for it is only that Number or 9's taken a number of times. But, 2. If each of them is equal to a Number of 9's, and some lesser Number over, then let us represent them thus; Let one be $A + b$, where A represents any Number of 9's, and b the Number over: Let the other be $N + c$, where N is any Number of 9's, and c the Number over. But now, by *Lem. 4.* the Product of these *Factors* is equal to the Sum of the Products of all the Parts of the one by all the Parts of the other; and so the Product is here $A \times N$, $+ A \times c$, $+ N \times b$, $+ b \times c$. But the first three Products are each a Number of 9's, because one of their *Factors* is so; therefore these being cast away, there remains only $b \times c$. And if the 9's are also cast out of this, the Excess is the Excess of 9's in the total Product; but b, c are the Excess in the *Factors*, and $b \times c$ their Product: therefore the Rule is true.

SCHOLIUM. The Objection made to this *Proof* is the same as that mentioned already against the like *Proof* for *Addition*; and therefore the same Answer serves here.

§. 2. Containing

PARTICULAR METHODS of working MULTIPLICATION in certain Cases; whereby it is either contracted into a shorter Work than the general Rule, or made easier and more certain with as large a Work; and, in some Cases, with a little more Work.

THE first five CASES are such wherein the very same Operation is performed as by the General Rule; only the Trouble and Time of writing down a great many Figures is saved.

CASE I. When one of the *Factors* has any Figure whatever in the place of Units, and 1 in all the rest, as 16, 114, 1118; make that the *Multiplier*, and the Product may be got all at once thus;

(1.) Suppose the number of Places of the *Multiplier* do not exceed those of the *Multiplicand*, then multiply by the Figure in the Units place of the *Multiplier*, and with every Product, till you come to the Product of that Figure that stands over the last 1 of the *Multiplier*, (i. e. till you have multiplied as many Figures as the Number of Places in the *Multiplier*) add in all the preceding Figures on the right of the *Multiplicand*; to the next Product after this, add in all the preceding except the first, (or Units place) and at every

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succeeding Product exclude always one more in order from the right, [and when you come to these exclusions, it will be fit to set a Point over the Figure which is to be excluded at the next Operation,] viz. the remotest on the right of these that were taken in at the last step. Write down at every step what's under or over 10's, and carry the 10's to the next step; and when you have gone thro' all the Figures of the *Multiplicand*, to the number of 10's carried from the last step add in all the Figures of the *Multiplicand* after the last Point, pointing also the last you take in at this step; then take in all from the last Point, and so on till you take in the very last Figure alone; and thus you have the true Product sought; as in Examples 1, 2, 3.

(2.) If the Places of the *Multiplier* exceed in Number those of the *Multiplicand*, do all as in the former Case; only, when all the Figures of the *Multiplicand* are multiplied, the whole of them must be summ'd and taken in as oft, and once more, as the difference betwixt the Number of Places of the *Multiplicand*, and the Number of 1's in the *Multiplier*, (i. e. plainly till you have got a product Figure under every 1 of the *Multiplier*;) then beginning still at the last place, take in one place fewer, (as before) till the last place is taken in alone. See Example 4.

The Reason of this Practice will easily appear by comparing a few Examples wrought this way, and also the common way.

Example 1.

$$\begin{array}{r} \text{~~~~~} \\ 4658 \\ \underline{16} \\ 74528 \end{array} \qquad \begin{array}{r} \text{~~~~~} \\ 4658 \\ \underline{16} \\ 27948 \\ 4658 \\ \underline{74528} \end{array}$$

Example 2.

$$\begin{array}{r} \text{~~~~~} \\ 5276 \\ \underline{114} \\ 601164 \end{array} \qquad \begin{array}{r} \text{~~~~~} \\ 5276 \\ \underline{114} \\ 21104 \\ 5276 \\ 5276 \\ \underline{601164} \end{array}$$

Example 3.

$$\begin{array}{r} \text{~~~~~} \\ 263487 \\ \underline{1113} \\ 293261031 \end{array} \qquad \begin{array}{r} \text{~~~~~} \\ 263487 \\ \underline{1113} \\ 790461 \\ 263487 \\ 263487 \\ 263487 \\ \underline{293261031} \end{array}$$

Example 4.

$$\begin{array}{r} \text{~~~~~} \\ 847 \\ \underline{11113} \\ 9412711 \end{array} \qquad \begin{array}{r} \text{~~~~~} \\ 847 \\ \underline{11113} \\ 2541 \\ 847 \\ 847 \\ 847 \\ 847 \\ \underline{9412711} \end{array}$$

The Operations of these Examples are thus; viz.

Examp. 1. $6 \times 8 = 48$, which is 8 and carry 4; then $6 \times 5 (= 30) + 4$ carried $(= 34) + 8$ (the next Figure on the right) $= 42$; which is 2, and carry 4. Then $6 \times 6 (= 36) + 4$ carried $(= 40) + 5$ (on the right) $= 45$; which is 5, and carry 4. Then $6 \times 4 (= 24) + 4$ carried $(= 28) + 6$ (on the right) $= 34$; which is 4, and carry 3; then 3 (carried) $+ 4$ (the last Figure) $= 7$.

Examp. 2. $4 \times 6 = 24$; which is 4, and carry 2: then $4 \times 7 (= 28) + 2$ carried $(= 30) + 6$ (on the right) $= 36$; which is 6, and carry 3: then $4 \times 2 (= 8) + 3$ carried $(= 11) + 7 + 6$ (the two next on the right) $= 24$; which is 4, and carry 2: then $4 \times 5 (= 20) + 2$ carried $(= 22) + 2 + 7$ (on the right) $= 31$; which is 1 and carry 3: then 3 carried

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ried $+5+2$ (on the right) $=10$; which is 0, and carry 1: then 1 (carried) $+5$ (on the right) $=6$.

Examp. 3. Here I shall spare repeating what is set down and carried; or the words *carried*, and *on the right*, because you know how to supply them. The Work is thus: $3 \times 7 = 21$; then $3 \times 8 (=24) + 2 + 7 = 33$; then $3 \times 4 (=12) + 3 + 8 + 7 = 30$; then $3 \times 3 (=9) + 3 + 4 + 8 + 7 = 31$; then $3 \times 6 (=18) + 3 + 3 + 4 + 8 = 36$; then $3 \times 2 (=6) + 3 + 6 + 3 + 4 = 22$; then $2 + 2 + 6 + 3 = 13$; then $1 + 2 + 6 = 9$; then (0 carried) $+2$.

Examp. 4. Thus $3 \times 7 = 21$; then $3 \times 4 (=12) + 2 + 7 = 21$; then $3 \times 8 (=24) + 2 + 4 + 7 = 37$; then $3 + 8 + 4 + 7 = 22$; then $2 + 8 + 4 + 7 = 21$; then $2 + 8 + 4 = 14$; then $1 + 8 = 9$.

SCHOLIUM. To compare these Operations with the same Examples at large, observe what Figures of the *Multiplicand* (as it stands written down for every 1 in the *Multiplier*) stand under each Figure of the first product Line; and these shew the reason of taking in the Figures on the Right-hand of the *Multiplicand*, and how this Rule was invented.

I advise a Learner to make himself familiar with the Practice of Examples like the first; because they occur frequently in common Business.

Also if the *Multiplier* has but two Places, tho' 2 is in the second place, the Product may easily be made all at once, by taking in with every Product made by the place of Units, double the Figure on the Right-hand. Practice will make this easy, and it will be very useful. The like Method may be used whatever Figure is in the second Place, (by taking in with every Product as many times the preceding Figure:) but the greater that Figure is, it is the more difficult; and I would only recommend the Practice for 2; unless the *Multiplicand* have not above 3 or 4 Places, and these also small Figures; for then we may use this Method with 3, 4, or 5 in the second Place.

$\begin{array}{r} 546 \\ 28 \\ \hline 15288 \end{array}$ The Example annex'd is thus done; $8 \times 6 = 48$; then $8 \times 4 (=32) + 4 + 12 = 48$; then $8 \times 5 (=40) + 4 + 8 = 52$; then $5 + 10 = 15$.

Note, If 1 is in all the Places, the Practice is so much the easier, without altering the Rule.

CASE II. If one of the *Factors* has a significant Figure in the Place of Units, 1 in the highest Place, and 0's betwixt them; make that in Units place the *Multiplier*, and work thus.

1. Suppose the places of the *Multiplier* do not exceed in number those of the *Multiplicand*; then multiply by the first Figure; and when you are come to the Figure of the *Multiplicand* that stands over the 1 in the *Multiplier*, (*i. e.* having made as many Products as there are Places in the *Multiplier*) with that Product take in the first Figure of the *Multiplicand*; and with every succeeding Product add in the next Figure in order, (and it will be convenient to put a Point over every Figure when taken in, that you may more readily know what is to be next taken in.) After the last Figure of the *Multiplicand* is multiplied, set down all the remaining Figures of the *Multiplicand* that are on the left of the Figure last taken in, after having added to them what was carried from the last Product.

The Reason of this is obvious in the following Examples.

Example 1.

$$\begin{array}{r} \dots \quad 57648 \\ 57648 \\ 103 \\ \hline 5937744 \end{array}$$

Example 2.

$$\begin{array}{r} 4653 \\ 1004 \\ \hline 4671612 \end{array}$$

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2. Suppose the Places in the *Multiplier* exceed those in the *Multiplicand*; then if they exceed only by one, there is no place for a *Contraction*, (*Examp. 3, 4.*) But if there is only one Place more, it may receive a small *Contraction* thus; Add what's carried at the last *Product* to the whole *Multiplicand*, and set down the Sum on the left of the Figures already set down; (*Examp. 5.*)

Example 3.

$$\begin{array}{r} 3207 \\ 100005 \\ \hline 220716035 \end{array}$$

Example 4.

$$\begin{array}{r} 468 \\ 1000007 \\ \hline 468003276 \end{array}$$

Example 5.

$$\begin{array}{r} 8674 \\ 10006 \\ \hline 86792044 \\ 8674 \\ \hline 86792044 \end{array}$$

Observe, If there are any more than one 1 upon the Left-hand of the o's, this will be but a Mixture of this and the preceding Case, to be done thus; *viz.* After you begin to take in the Figures of the *Multiplicand*, take them in gradually, first one, then two, and so on, till you take in as many as the number of 1's in the *Multiplier*, (beginning still at the first Figure of the *Multiplicand*.) and after that (taking in still the same Number of Figures, begin one place nearer the Left-hand; and when you have not as many Figures to take in, take in all you have. The following Examples, without any more Words, will sufficiently explain this.

Example 6.

$$\begin{array}{r} 853467 \\ 11004 \\ \hline 9291550868 \\ 3413868 \\ 853467 \\ 852467 \\ \hline 9291550868 \end{array}$$

Example 7.

$$\begin{array}{r} 67853945 \\ 111007 \\ \hline 7532262872615 \end{array}$$

Example 8.

$$\begin{array}{r} 4632 \\ 111004 \\ \hline 514170528 \end{array}$$

Example 9.

$$\begin{array}{r} 463 \\ 11004 \\ \hline 5094852 \end{array}$$

Example 10.

$$\begin{array}{r} 463 \\ 110004 \\ \hline 50931852 \end{array}$$

Example 11.

$$\begin{array}{r} 463 \\ 1100004 \\ \hline 50371852 \end{array}$$

CASE III. When one of the *Factors* has any Figure greater than 1 in the highest Place, and 1 in all the other Places make that Figure the *Multiplier*: then work thus, *viz.* Take the Figure in Units place of the *Multiplicand*, and set it down; next take the Sum of the first two places, and then the Sum of the first three Places, and so on (beginning still at the first Place) till you have made as many such Operations as the Number of times that 1 is in the *Multiplier*, (and if there are not as many Figures in the *Multiplicand* as to have a Figure more to take in at every Operation, you must continue to take in the whole *Multiplicand* till your number of Operations are compleat) still writing down the Sums as in *Addition*, and carrying the 10's. After this, take the Figure in the highest Place, and by it multiply the whole *Multiplicand* in order; taking into the first *Product* what was carried from the preceding Operations; and to this and each other *Product* add the Sum of

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of as many of the next following Figures on the left as the Number of 1's in the *Multiplier*; and if there are not as many, add all that are.

Note, It will be convenient to set the highest Place of the *Multiplier* under the first of the *Multiplicand*.

Example 1.

$$\begin{array}{r} 4768 \\ 41 \\ \hline 195488 \end{array} \quad \begin{array}{r} 4768 \\ 41 \\ \hline 4768 \\ 19072 \\ \hline 195488 \end{array}$$

Example 2.

$$\begin{array}{r} 7236 \\ 4111 \\ \hline 29747196 \end{array} \quad \begin{array}{r} 7236 \\ 4111 \\ \hline 7236 \\ 7236 \\ 7236 \\ 28944 \\ \hline 29747196 \end{array}$$

Example 3.

$$\begin{array}{r} 86 \\ 4111 \\ \hline 353546 \end{array} \quad \begin{array}{r} 86 \\ 4111 \\ \hline 86 \\ 86 \\ 86 \\ 344 \\ \hline 353546 \end{array}$$

CASE IV. When one of the *Factors* has 1 in the first Place, any Figure in the highest Place, and o's betwixt them, make that Figure the *Multiplier*, and work thus; *viz.* Write down as many of the first places of the *Multiplicand* as are in Number one less than the Places in the *Multiplier*, (*Examp. 1, 2, 3.*) and if there are not as many, make it up with o's set on the left, (*Examp. 4.*) After this, multiply by the last Figure of the *Multiplier*, taking with every Product the Figure of the following Place but one or two, &c. according to the Number of o's in the *Multiplier*, (and set a Point over the Figures taken in, which will be a guide to the next) and you have the Product sought.

Examp. 1.

$$\begin{array}{r} 678 \\ 301 \\ \hline 204078 \end{array} \quad \begin{array}{r} 678 \\ 301 \\ \hline 678 \\ 2034 \\ \hline 204078 \end{array}$$

Examp. 2

$$\begin{array}{r} 34675 \\ 3001 \\ \hline 104059675 \end{array} \quad \begin{array}{r} 34675 \\ 3001 \\ \hline 34675 \\ 104025 \\ \hline 104059675 \end{array}$$

Examp. 3.

$$\begin{array}{r} 496 \\ 8001 \\ \hline 3968496 \end{array}$$

Examp. 4.

$$\begin{array}{r} 7263 \\ 400001 \\ \hline 2905207263 \end{array}$$

CASE V. When one of the *Factors* consists of the same Figure in all its Places (as 66, or 444,) make that the *Multiplier*, and work thus; *viz.* Multiply by that Figure, and out of this Product make up the total Product in this manner: Begin at the Place of Units, and first take one Figure, then two, then three, &c. (setting down the Sums under or over 10's, and carry the 10's;) repeating the Operation still from the first Place, as oft as the Number of Places in the *Multiplier*; and if there is not another Figure to take in at every Operation, you must continue to take in the whole *Multiplicand*, so oft till your Number of Operations be finished. Then begin at the second place, and third place, &c. successively; taking in from each as many on the left as the Number of Places of the *Multiplier*, as long as you can find as many; and when they fail, take in all that remains till the last Figure is taken in alone; (minding always to carry the 10's from every Operation to the next.)

I

Examp.

Example 1.

$\begin{array}{r} 8739 \\ 444 \\ \hline 34956 \text{ Prod. by 4} \\ 3880116 \text{ Total.} \end{array}$	$\begin{array}{r} 8739 \\ 444 \\ \hline 34956 \\ 34956 \\ 34956 \\ \hline 3880116 \end{array}$
---	--

Example 2.

$\begin{array}{r} 46 \\ 33333 \\ \hline 138 \\ \hline 1533318 \end{array}$	$\begin{array}{r} 46 \\ 33333 \\ \hline 138 \\ 138 \\ 138 \\ 138 \\ 138 \\ \hline 1533318 \end{array}$
--	--

SCHOLIUM. If the *Multiplier* consists of 1 in all its places, the Practice is the same; only we have not any previous Multiplication, the second part of the Work being made upon the *Multiplicand* itself.

COROL. When the *Multiplier* consists of any other Figures than 1, we may do the Work thus: First find the Product, as if the *Multiplier* consisted of as many 1's, and then multiply this Product by the Figure of the given *Multiplier*; so the preceding *Example* will stand thus:

$$\begin{array}{r} 8739 \times 444 \\ \hline 970029 \text{ Prod by 111.} \\ 4 \\ \hline 3880116 \end{array}$$

The Reason of this you'll find at *Case 9.* joined with this, that $444 = 111 \times 4$.

CASE VI. If the *Multiplier* is a Number which has 9 in all its places, as 9, 99, 999, &c. set as many 0's on the Right-hand of the *Multiplicand*, and subtract it from itself so increased: the Remainder is the Product sought, as in these Examples.

Example 1.

$$468 \times 9 = 4212$$

Operation.

$$\begin{array}{r} 4680 \\ 468 \\ \hline 4212 \end{array}$$

Example 2.

$$3726 \times 99 = 368874$$

$$\begin{array}{r} 372600 \\ 3726 \\ \hline 368874 \end{array}$$

Example 3.

$$7568 \times 999 = 7560432$$

$$\begin{array}{r} 7568000 \\ 7568 \\ \hline 7560432 \end{array}$$

The Reason of this Rule is obvious, for by the 0's prefix'd to the *Multiplicand*, it is multiplied by a Number which exceeds the given *Multiplier* by 1, (so $10 - 1 = 9$, $100 - 1 = 99$, and so on.) Wherefore the *Multiplier* being subtracted from this Product, the Remainder must be the true Product sought.

Observe, This Subtraction may easily be perform'd without the trouble of writing the *Multiplicand* oftner than once, by imagining the 0's that ought to be prefix'd: Thus, take the first Figure of the *Multiplicand* from 10, then the next Figure increased by 1, from 10, and so on, till you have made as many Subtractions from 10, as the Number of 9's in the *Multiplier*: then you subtract from the Figures of the *Multiplicand* itself; and when there are no more Figures to subtract, write down all the remaining Figures from which no Subtraction has been made, after subtracting 1 from them when 10 was borrowed in the preceding Subtraction. The preceding *Examples* sufficiently shew how this is to be done.

Observe again, To multiply any Number consisting all of 9's by itself; the Product has 1 in the place of Units, then after it a Number of 0's fewer by one than there are 9's in the Number multiplied, then 8, and lastly, after it as many 9's as there are 0's before it.

Ex. $999 \times 999 = 998001$, and $99999 \times 99999 = 9999800001$.

The

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The *Reason* of this will appear to be universal, by considering one *Example* done in the manner of the preceding Rule; thus,

$$\begin{array}{r} 9999900000 \\ 99999 \\ \hline 9999800001 \end{array}$$

CASE VII. To multiply any Number by 5; multiply it first by 10, *i. e.* set (or suppose) 0 before it; then take the half of it.

Ex. 1. 6458×5
 $\begin{array}{r} 6458 \\ \times 5 \\ \hline 32290 \end{array}$ Prod.

Ex. 2. 7287×5
 $\begin{array}{r} 7287 \\ \times 5 \\ \hline 36435 \end{array}$ Prod.

SCHOLIUM. I have here proposed a Division, tho' that Rule is not yet taught: But it was fit to put all together that relates to Multiplication, tho' the Learner should refer this Case till he has Division; yet I think it will be easy to see how any Number is halved, by considering these two

Examples. And as to the Use of this Rule, it's certainly easier than multiplying by 5, tho' that is easy itself.

GENERAL COROLLARY to the preceding 7 Cases. If the Parts of the *Multiplier* coincide with any of the preceding Cases, we may apply them separately: As in these Examples.

Ex. 1. 7468
 $\begin{array}{r} 7468 \\ \times 159 \\ \hline 67212 \\ 112020 \\ \hline 1187412 \end{array}$

Ex. 2. 367845
 $\begin{array}{r} 367845 \\ \times 4118 \\ \hline 6621210 \\ 15081645 \\ \hline 1514785710 \end{array}$

For Ex. 1. Work by 9 (as in Case 6) then for 15 (as Case 1.)

For Ex. 2. Work with 18, and 41 (by Case 1. and 3.)

CASE VIII. If one of the *Factors* consists of two Places, and is equal to the Product of any two Digits, as $28 = 4 \times 7$; multiply by one of these Digits, and the Product found by the other; and the last Product is that sought.

Ex. 7264×28
 $\begin{array}{r} 7264 \\ \times 4 \\ \hline 29056 \\ \times 7 \\ \hline 203392 \end{array}$ Prod.

The Reason of this is obvious, for 7 times 4 times is 28 times. Or may be deduced from *Lemma 3. Corol.* For having multiplied by 4, which is only the 7th part of 28, I must multiply again by 7 the Denominator of the Part, to make the true Product by the whole 28.

SCHOLIUM. If the *Multiplier* is equal to the *Product* of any three or more Digits; we may take the same Method, by multiplying continually by all these Digits: But this will not in every Case afford any Compend; nor will it always appear easily what Digits will

produce the given Number: However the Reason of the Practice is evident from *Lem. 3. Cor.* For $7486 \times 8 \times 9 \times 6 = 8 \times 9 \times 6 \times 7486 = 432 \times 7486$, because $8 \times 9 \times 6 = 432$.

Ex. $7486 \times 432 (= 8 \times 9 \times 6)$
 $\begin{array}{r} 7486 \\ \times 8 \\ \hline 61408 \\ \times 9 \\ \hline 553392 \\ \times 6 \\ \hline 3320352 \end{array}$ Product.

CASE IX. In Multiplication of great Numbers, one part of the hazard of erring proceeds from the too great Distance betwixt the Figures of the several Product Lines and the corresponding Figures of the *Multiplicand*; and also their not standing directly under one another, so that we must always look athwart from the one to the other.

In many Cases, tho' we cannot contract the Work, either as to the Operation or Number of Figures written, yet we may make it more simple and easy, by help of Addition or Subtraction, or more simple Multiplication; and sometimes with these Methods we may mix

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mix some of the former. The following Examples will sufficiently instruct you how to do the like in other Cases: For the different Circumstances of the *Multiplier* makes the Variety unlimited, and therefore there can be no general Rule.

<p>Ex. 1.</p> $\begin{array}{r} 648 \\ 76 \\ \hline 3888 \\ 4536 \\ \hline 49248 \end{array}$	<p>Ex. 2.</p> $\begin{array}{r} 467 \\ 78 \\ \hline 3736 \\ 3269 \\ \hline 36426 \end{array}$	<p>Or thus.</p> $\begin{array}{r} 467 \\ 78 \\ \hline 3269 \\ 3736 \\ \hline 36426 \end{array}$	<p>Ex. 3.</p> $\begin{array}{r} 57684 \\ 63 \\ \hline 173052 \\ 346104 \\ \hline 3634092 \end{array}$	<p>Ex. 4.</p> $\begin{array}{r} 974 \\ 862 \\ \hline 1948 \\ 5844 \\ 7792 \\ \hline 839588 \end{array}$
<p>Ex. 5.</p> $\begin{array}{r} 5689 \\ 286 \\ \hline 11378 \\ 45512 \\ 34124 \\ \hline 1627054 \end{array}$	<p>Ex. 6.</p> $\begin{array}{r} 4956 \\ 168 \\ \hline 39648 \\ 79296 \\ \hline 832608 \end{array}$	<p>Or thus.</p> $\begin{array}{r} 4956 \\ 168 \\ \hline 79296 \\ 39648 \\ \hline 832608 \end{array}$	<p>Ex. 7.</p> $\begin{array}{r} 5648 \\ 459 \\ \hline 50832 \\ 254160 \\ \hline 2592432 \end{array}$	<p>Ex. 8.</p> $\begin{array}{r} 724689 \\ 545 \\ \hline 3623445 \\ 26611005 \\ \hline 398955505 \end{array}$

The Operation of these Examples is thus:

Ex. 1. After multiplying by 6, I add that Product to the *Multiplicand*, instead of multiplying by 7.

Ex. 2. After multiplying by 8, I subtract the *Multiplicand* out of this Product, instead of multiplying by 7. Or, according to the other Method, I first multiply by 7, and to this Product add the *Multiplicand*. The reason of placing the two Lines as you see them stand here, is obvious.

Ex. 3. After multiplying by 3, I double this Product for 6.

Ex. 4. I multiply by 2, then multiply this Product by 3 (for 6) then add these two Lines (the first Figure of the one to the first of the other, and so on) for 8.

Ex. 5. I multiply by 2, this Product by 4, (for the 8) and then subtract the first Line from the second (for the 6) or multiply the first Line by 3.

Ex. 6. I multiply by 8, and double this Product (for the 16,) Or multiply by 16 and halve this Product (for the 8.)

Ex. 7. I multiply by 9, and because $9 \times 5 = 45$, I multiply the last Product by 5, either the common way, or by Case 7.

Ex. 8. I multiply by 5 (Case 7.) for the last 5; and this Product by 9 (Case 6.) for the 45.

An UNIVERSAL METHOD for all Cases; whereby, tho' there is no Contraction, and even some more to do, yet it makes the Work so easy, that there is no time lost, at least in large Examples, and more Certainty in the Operation. Thus:

Write down the *Multiplicand*, then double it; add this Sum to the *Multiplicand*, and this again, and so on, every Sum to the *Multiplicand*, till you have nine Numbers. And it's plain that thus you have a Table of the Products of the *Multiplicand* by all the Digits; made up by a very simple and easy Operation: And then you have no more to do, but transfer your several Products out of this Table, and sum them up.

TABLE.

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T A B L E.

1	467853798
2	935707596
3	1403561394
4	1871415192
5	2339268990
6	2807122788
7	3274976586
8	3742830384
9	4210684182

Example.

467853798
6839754
1871415192
2339268990
3274976586
4210684182
1403561394
3742830384
2807122788
3200004886285692

SCHOLIUMS.

1. This Method is universal; but we need not apply it in every Case, for that would not always be best: But in such Examples as this, I think the Ease and Readiness with which it's done, does more than save the time spent in making the Table; with this Advantage, that the Work is perform'd with much more Certainty, because it's more simple.

2. Again this may be contracted in many Cases: for there is no necessity always to make the Table for all the nine Digits. And it may happen, that by help of some of the preceding Methods, we can as easily make a Table for few, or no more than we have use for in the *Multiplier*; nor is it any great matter in what order they stand in the Table.

Ex. 1. 78659 by 6897. In making this Table, I first take 3; then double its Product for 6; and do the rest by the common way.

Table for Ex. 1.

1	78659
3	227977
6	455954
7	534613
8	613272
9	691932

Ex. 2. 783596 by 3856. Table for Ex. 2.

First I multiply by 3, then by 5, (as in Case 7.) then add these Products for 8, then subtract the first from this for 7.

1	783596
3	2350788
5	3917980
8	6268768
7	5485172

MULTIPLICATION by NEPER'S RODS.

The great and everlasting Honour of our Country, the Lord Neper, considering the vast Advantage of the preceding Method of *Tabulating* the *Multiplicand*, to make it yet more easy, contrived the following Machine, for the more certain and ready way of making the Table: To understand which, we must set before us what they call,

PYTHAGORAS's Table of Multiplication.

1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

The Construction of this Table is the same as that shewn in *Problem 1.* with this difference, that here there is a compleat Column of Products from every Digit on the head, to 9 times that Digit; so that either of the Factors may be found on the head, or on the left side.

Now, suppose this Table to be made upon a Plate of Metal, Ivory, Wood, or Past-board; and then conceive the several Columns (standing downwards from the Digits on the head) to be cut asunder, and these are what we call *Neper's Rods*, for *Multiplication*. But then there must be a good number of each; for as many

H

times

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times as any Figure is in the *Multiplicand*, so many Rods of that Species, (*i. e.* with that Figure on the Top of it) must we have; tho' 6 Rods of each Species will be sufficient for any Example in common Affairs. There must also be as many Rods of 0's. But before we explain the way of using these Rods, there is another thing to be known, *viz.* That the Figures on every Rod are written in an Order different from that in the Table; *Thus*, The little square Space or Division in which the several Products of every Column are written, is divided into two Parts by a Line a-cross from the upper Angle on the Right to the lower on the Left; and if the Product is a Digit, it is set in the lower Division; if it has two Places, the first is set in the lower, and the second in the upper Division; but the Space on the Top is not divided. Also there is a Rod of Digits not divided, which is called the *Index-Rod*; and of this we need but one single Rod. See here the Figure of all the different Rods, and the Index, separate from one another.

NEPER'S RODS.

Index Rod.	1	1	2	3	4	5	6	7	8	9	0
2	2	4	6	8	10	12	14	16	18	0	0
3	3	6	9	12	15	18	21	24	27	0	0
4	4	8	12	16	20	24	28	32	36	0	0
5	5	10	15	20	25	30	35	40	45	0	0
6	6	12	18	24	30	36	42	48	54	0	0
7	7	14	21	28	35	42	49	56	63	0	0
8	8	16	24	32	40	48	56	64	72	0	0
9	9	18	27	36	45	54	63	72	81	0	0

USE of the RODS.

Lay down first the Index-Rod; on the Right of it, set a Rod on whose Top is the Figure in the highest Place of the *Multiplicand*; next to this again set the Rod on whose Top is the next Figure of the *Multiplicand*; and so on in order to the first Figure: Then is your *Multiplicand* tabulated for all the 9 Digits; for in the same Line of Squares standing against every Figure of the Index-Rod, you have the Product of that Figure; and therefore you have no more to do but transfer the Products, and sum them.

But in taking out these Products from the Rods, the Order in which the Figures stand obliges you to a very easy and small Addition, thus; Begin to take out the Figure in the lower Part (or *Units* place) of the Square of the first Rod on the Right; add the Figure in the upper Part of this Rod to that in the lower Part of the next, and so on; which may be done as fast as you can look upon them. To make this Practice as clear as possible, take this Example.

Examp.

The Rods set together for the Number 4768.

1	4	7	6	8
2	$\frac{8}{1}$	$\frac{4}{2}$	$\frac{1}{2}$	$\frac{1}{6}$
3	$\frac{1}{2}$	$\frac{2}{1}$	$\frac{1}{8}$	$\frac{2}{4}$
4	$\frac{1}{6}$	$\frac{2}{8}$	$\frac{2}{4}$	$\frac{3}{2}$
5	$\frac{2}{0}$	$\frac{3}{5}$	$\frac{3}{0}$	$\frac{4}{0}$
6	$\frac{2}{4}$	$\frac{4}{2}$	$\frac{3}{6}$	$\frac{4}{8}$
7	$\frac{2}{8}$	$\frac{4}{9}$	$\frac{4}{2}$	$\frac{5}{6}$
8	$\frac{3}{2}$	$\frac{5}{6}$	$\frac{4}{8}$	$\frac{6}{4}$
9	$\frac{3}{6}$	$\frac{6}{3}$	$\frac{5}{4}$	$\frac{7}{2}$

Examp. Multiply 4768 by 385.

Against 5 in the Index I find this } 23840
 Number, according to the Rule, }
 Against 8 this Number, — — 38144
 Against 3 this Number, — — 14304
 Total Product, 1835680

To make the Use of the *Rods* yet more regular and easy, they are kept in a flat square Box, whose Breadth is that of ten Rods, and the Length that of one Rod; as Thick as to hold six, (or as many Rods as you please;) the Capacity of the Box being divided into ten Cells, for the different Species of Rods. When the Rods are put up in the Box, (each Species in its own Cell, distinguished by the first Figure of the Rod set before it on the Face of the Box near the Top,) as much of every Rod stands without the Box as shews the first Figure of that Rod: Also upon one of the flat Sides, without and near the Edge upon the Left-hand, the Index-Rod is fixed; and along the Foot there is a small Ledge; so that the Rods when applied are laid upon this Side, and supported by the Ledge, which makes the Practice very easy. But in case the *Multiplicand* should have more than 9 Places, that upper Face of the Box may be made broader.

Some make the *Rods* with four different Faces, and Figures on each, for different purposes. But I have explained what is necessary for *Common Multiplication*, and shall leave it.

A NEW METHOD by a small Moveable TABLE.

To Those who want *Rods*, I propose the following Methods, which, with the help of *Case VI.* and *VII.* and some Hints given in *Case X.* may, I believe, be near as easy and expeditious as the *Rods*. Thus:

1. Make a Table of the *Multiplicand* only for the Numbers 1, 2, 5; (using *Case VII.* for 5.) and make it upon a bit of loose Paper, that it may be always applied directly and immediately over the Place where every particular Product is to be written down, (for much of the difficulty lies, as I have already observed, in the Distance and cross Position of the *Multiplicand* to the several Products,) and out of this small Table find your Product thus:

Suppose for a Multiplicand 685497.

T A B L E.

1	685497
2	1370994
5	3427485

When the Figure of the Multiplier is 2 or 5, here you have the Products; then for 3 add 2 to 1, (*i.e.* the Numbers against 2 and 1;) for 4, double the Number against 2. For 6, add 5 and 1; or multiply 2 by 3. For 7 add 5 and 2. For 8 add 5, 2 and 1; for 9, use the Method of *Case VI.*

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2. We

2. We may also make the Table for 3 and 5, as here: And then for 2, double 1; for 4, add 3 and 1; for 6, double 3, or add 5 and 1; for 7, double 3 and add 1; that is, to every Product add the Figure of the *Multipl- cand.* For 8, add 5 and 3; for 9, use *Case VI.*

1	685497
3	2056491
5	3427485

1	685497
3	2056491
-	1-084-9

1	685497
2	1370994
4	2741988

1	685497
3	2056491
4	2741988

3. Or also make the Table for 3 and 7: And to get the 7, I multiply 3 by 2, and add 1. Then in using the Table, to get the Product by 4, add 2 to 1. For 5, use *Case VII.* For 6, double 2; for 8, add 1 and 7; for 9, use *Case VI.* or multiply 3 by 3.

4. Or we may make the Table for 1, 2, 4. Then for 3, add 1 and 2; for 5, use *Case VII.* or add 1 and 4. For 6, add 2 and 4; for 7, add 1, 2, and 4; for 8, double 4; for 9, use *Case VI.*

5. Or lastly, make the Table for 1, 3, 4. Then for 2, double 1; for 5, use *Case VII.* for 6, double 3; for 7, add 4 and 3; for 8, double 4; for 9, use *Case VI.*

Use any of these Tables you please; and for different Multipliers, one of them may, perhaps, be preferable to another. But if all the 9 Digits are in any Multiplier, it is indifferent which of them you chuse; tho' I think the third Method has the Advantage.

CHAP. VI.

DIVISION of Abstract Whole NUMBERS.

DEFINITION.

DIVISION findeth how oft one Number is contained in another.

The Number divided, (or which is consider'd as the containing Number) is called the *Dividend*; the Number dividing, (or which is considered as contained) is called the *Divisor*; and the Number sought, is called the *Quotient*, or *Quote*, (from *Quoties, How oft*;) because it shews the *how oft* sought. *Examp.* If we enquire how oft 3 is contained in 12, the Answer is 4 times: And 12 is the *Dividend*; 3 the *Divisor*; 4 the *Quote*.

SCHOLIUMS.

1. Every greater Number is not a *Multiple* of every lesser: therefore when a greater Number is proposed to be divided by a lesser, and is not a *Multiple* of it, this Operation finds how oft the *Divisor* is contained in the *Dividend*; and also what remains after the *Divisor* is taken out of the *Dividend* as oft as possible: So that in some Cases there are four Numbers concerned in *Division*; viz. the *Divisor*, *Dividend*, *Quote*, and *Remainder*. *Examp.* 3 is contained in 14, 4 times, and 2 remains.

2. As *Multiplication* is only a compendious *Addition*; so is *Division* only a compendious *Subtraction* of one Number out of another as oft as possible: For it is plain, that as oft as any Number is contained in another, so oft precisely it can be taken out of it; so

so that if we find how oft the lesser can be taken out of the greater, we thereby find how oft it is contained in it : but to find how oft it can be taken out, is plainly the work of Subtraction; for by taking the lesser out of the greater, and the same lesser out of the first Remainder, and out of every Remainder successively till the Remainder be 0, or less than the Subtractor, we have found what is required is *Division*; the Number of Sub-

tractions being the *Quote*. So in these Examples, we find that 3 is contained in 12, 4 times; because it can be taken out of it 4 times, and 0 remains. And 3 is contained in 14, 4 times, and 2 remains; because being 4 times subtracted, there remains 2.

But tho' the thing sought in *Division* may be found by Subtraction, yet it would be intolerable Labour in most Cases; and therefore the following Rules of *Division* are contrived, which do that by a few easy steps which cannot be done all at once, and would be too tedious to do by Subtraction.

Again *observe*, That tho' there are here also, as in the preceding Operations, some more simple Cases, the ready performance of which is useful in more complex Cases: yet in most Cases we are left to guess at the Answer in the several steps of the Work; with this help only,

that we have a certain Rule for proving the Number guessed to be right, and when it is wrong, how to come nearer to it at the next guess, till at last we find it; as you will presently learn.

3. In all Cases where there is a Remainder in the *Division*, or when the *Dividend* is not a Multiple of the *Divisor*, the Number called the *Quote* is the direct and proper Answer to that Question, How oft is the *Divisor* contained in the *Dividend*? yet the Remainder may be brought in fractionally as a Part of the *Quote*, making the Remainder the Numerator, and the *Divisor* the Denominator of a Fraction: And then we may say, That the *Dividend* contains the *Divisor* so many times as the *Integral Quote* expresses, and such a Part or Parts of a time, (*i. e.* such a Part or Parts of the *Divisor*), as that Fraction expresses. Thus, for Example, 3 is contained in 14, 4 times, and 2 remains; which being $\frac{2}{3}$ Parts of 3, we may say, that 14 contains 3, 4 times and $\frac{2}{3}$ Parts of a time; *i. e.* that it contains 4 times 3, and $\frac{2}{3}$ Parts of 3: And this mix'd Expression, $4 + \frac{2}{3}$; or thus, $4\frac{2}{3}$, may be called *The Complete Quote*, in distinction from the *Integral Quote*. There will be the same Reason in all Cases for completing the *Quote* by a Fraction made of the Remainder and *Divisor*, and understanding it as we have done in this Example. Thus, universally, if any Number A is contained in another B, a number of times expressed by D, with a Remainder R; then is the complete Quote $D\frac{R}{A}$: For any Number R being, the same as R times 1, and 1 being such a Part of any Number A, as that Number denominates; *i. e.* the $\frac{1}{A}$ Part; therefore R is $\frac{R}{A}$ Parts of A. Wherefore, if by the words, *How oft*, in the Definition, we mean how many times and parts of a time, (as now explained,) the *Integral Quote* must always have this Fraction, made of the Remainder and *Divisor*, annex'd to it as a part of the complete Answer of the Question.

Hence again *observe*, That in this sense, the *Dividend* may be a lesser Number than the *Divisor*; for tho' a lesser Number does not contain a greater, yet it contains a certain Fraction of it, which is what we call containing it a certain part or parts of a time; and the *Quote* is a Fraction whose Numerator is the *Dividend*, and its Denominator the *Divisor*. So 3 divided by 5, the *Quote* is $\frac{3}{5}$, signifying that 3 contains 5, $\frac{3}{5}$ parts of a time; or that it contains $\frac{3}{5}$ parts of 5.

4. But again, for the same Reason, (*viz.* that any Number A is equal to or contains $\frac{A}{B}$ Parts of any Number B) the complete *Quote* of any Number divided by any other Number

Ex. 1. Ex. 2.

$$\begin{array}{r} 12 \\ 3 \\ \hline 9 \\ 3 \\ \hline 6 \\ 3 \\ \hline 3 \\ 3 \\ \hline 0 \end{array} \quad \begin{array}{r} 14 \\ 3 \\ \hline 11 \\ 3 \\ \hline 8 \\ 3 \\ \hline 5 \\ 3 \\ \hline 2 \end{array}$$

Number may be indefinitely expressed by a Fraction whose Numerator is the Dividend, and its Denominator the Divisor. Thus 5 divided by 3, the *Quote* is $\frac{5}{3}$; and 3 divided by 5, the *Quote* is $\frac{3}{5}$. *Universally*, A divided by B, the *Quote* is $\frac{A}{B}$.

Now, as Fractions arise from *imperfect Division* when the Dividend is not a Multiple of the Divisor; so the Consideration of Fractions does necessarily begin with *Division of Whole Numbers*: For which Reason there are some things relating to Fractions must be explained in this Part; particularly I must here shew you, That the Fraction $\frac{A}{B}$ is in all Cases equal in Value to the complete Quote of A divided by B; and this being demonstrated, it will follow, that if we use such an Expression of a Quote in any Reasoning or Operation, instead of the more direct and immediate Expression of the Quote, which it is often very convenient to do, the Effect will be the same. And this I demonstrate thus:

1. If A is lesser than B, there is no other Quote. 2. If A is greater than B, then is $\frac{A}{B}$ an improper Fraction; and from the nature of such a Fraction, (as it is defined and explained in *Chap. I.*) this is manifest, That as oft as B is contained in A, so many Units (of that kind to which the Fraction refers) is the Fraction equal to, and to such a proper Fraction more, whose Numerator is the Remainder, (after B is taken out of A as oft as possible) and its Denominator B. For when the Numerator and Denominator are equal, the Value of the Fraction is 1; so $\frac{3}{3}$ Parts, or $\frac{B}{B}$ Parts of any thing, is equal to that Thing or Unit: Therefore, if B is contained in A, R times without a Remainder, $\frac{A}{B}$ is equal to R times 1, or R. And if there is a Remainder r , the Value of it is $R + \frac{r}{B}$, or $R \frac{r}{B}$.

The *Division* or *Quote* of two Numbers is also expressed by this Sign \div set betwixt them; the *Dividend* being set first, thus $5 \div 3$, signifies 5 divided by 3. $A \div B$ signifies A divided by B. It is also sometimes expressed by this Sign $)$, with the *Divisor* before the *Dividend*, thus $3) 5$. $A) B$.

Here then you have the *General Rule* of the LITERAL DIVISION.

COROLLARIES.

I. If the Integral Quote (or Number of times the whole Divisor is contained in the Dividend,) is multiplied into the Divisor, and to the Product be added the Remainder, (after the Divisor is taken out of the Dividend as oft as possible) the Sum is equal to the Dividend. For Example, 3 is contained 4 times in 12, and nothing remains; therefore 4 times 3 is 12. Again, 3 is contained 4 times in 14, and 2 remains; therefore 4 times 3, and 2 added, is 14: For $4 \times 3 = 12$, and $12 + 2 = 14$. *Universally*, if $A \div B = q$, and nothing remaining, then is $Bq = A$. But if r remains, then is $Bq + r = A$.

II. The Remainder in *Division* must always be less than the Divisor; for else the Divisor is not taken out of the Dividend as oft as possible: and this therefore is a Mark that the *Quote* is taken too small: As again, if the Product of the Divisor and Quote exceed the Dividend, it is a sign the Quote is taken too great. And hence, lastly, the Product of the Divisor and integral Quote is the greatest Multiple of the Divisor contained in the Dividend.

SCHOLIUMS.

I. We have here learnt a mutual *Proof* of *Multiplication* and *Division*, as these Operations are in their Effects directly opposite to one another. Thus, in *Multiplication*, if the Product is divided by any one of the Factors, the Quote is the other. And in *Division*, if the Divisor is multiplied by the integral Quote, and to the Product be added the Remainder, the Sum is the Dividend. Other *Proofs of Division* you will find afterwards.

II. If we compleat the Quote by a Fraction made of the Remainder and Divisor, then it is a general Truth, that the Divisor multiplied by the compleat Quote, produces the Dividend: For being multiplied by the integral Quote, the Product is the greatest Multiple of the Divisor contained in the Dividend; and multiplied by the Fraction, (*i. e.* such a Fraction of it being taken,) it produces the Remainder; thus, $14 \div 3 = 4 \frac{2}{3}$; then $4 \times 3 = 12$, and $\frac{2}{3}$ of $3 = 2$. And $12 + 2 = 14$. Universally, if $A \div B = q \frac{r}{B}$, then $A = Bq + \frac{r}{B}$ of $B = Bq + r$, as in *Corol. 1.* Observe also, That as in Whole Numbers, it is no matter which of two Numbers is the Multiplier; so, to multiply a Whole Number by a Fraction, or this by that, has the same Effect, as will be explained in *Book 2.* But in the Case now before us, the Reason is obvious. Thus to multiply B by $\frac{A}{B}$ is only taking $\frac{A}{B}$ Parts of B ; which is A . Again, to multiply $\frac{A}{B}$ by B , is taking B times, A times $\frac{1}{B}$; (for $\frac{A}{B}$ is A times $\frac{1}{B}$.) But B times $A = A$ times B , therefore B times A times $\frac{1}{B} = A$ times B times $\frac{1}{B} = A$ times 1 ; (for B times $\frac{1}{B} = 1$) or A . Wherefore to multiply the two Parts of the complete Quote by the Divisor, or to multiply the Divisor by these, the Sum of the Products is the Dividend. For $\frac{r}{B}$ of $B = B$ times $\frac{r}{B} = r$; and $Bq + r = A$. Or taking the complete Quote fractionally, then $\frac{A}{B}$ of $B = B$ times $\frac{A}{B} = A$.

P R O B L E M.

To Divide one Number by another.

CASE I. When the Divisor is a Digit, or single Figure, and the Dividend either a Digit, or a Number of two Figures, whereof that in the place of Tens is less than the Divisor.

Rule. Take such a Digit as, multiplied into the Divisor, will exactly produce the Dividend; but if there is not such a Digit, take the greatest, which multiplied into the Divisor, makes a Product less than the Dividend; that Digit is the integral Quote, and the Remainder (which must be less than the Divisor) set fractionally over the Divisor, compleats the Quote. The Reason of this Rule is in *Schol. 1.* preceding.

Examp. 1. $12 \div 3 = 4$; because $4 \times 3 = 12$. Examp. 2. $26 \div 4 = 6$, and 2 remains; so the complete Quote is $6 \frac{2}{4}$; for $6 \times 4 = 24$; then $24 + 2 = 26$. Examp. 3. $8 \div 5 = 1$, and 3 remains; so the complete Quote is $1 \frac{3}{5}$.

Whoever is familiar with the Table of *Multiplication*, can find at first hearing the Answer to any Example of this Case. Or we may take help of that Table, thus: Seek the Dividend in the Table; and if it is not there, seek the greatest Number which is less than it; the Digit in the same Line on the Side of the Table, or in the same Column on the head of the Table, is the Divisor, and the other is the integral Quote.

CASE II. To Divide any Number by another.

Rule. Set the Divisor on the left of the Dividend, as in the following Examples; then take as many Figures from the Left-hand of the Dividend as are in Number equal to the Places of the Divisor; and if these Figures, considered by themselves, make a Number less than the Divisor, take in one Figure more, [which will necessarily make a Number greater than the Divisor,] this we call the first *Dividual*, (or partial *Dividend*.) We are then to find how oft the Divisor is contained in this Dividual; and here it is that we are left in a great measure to guess at the Figure sought: But this we know, that the Quote cannot exceed 9, as shall be afterwards demonstrated. And then also (by *Corol. 2.* preceding,) it must

must be such, that the Product of the Divisor by it do not exceed the Dividual, (for then it is too great a Figure,) and the Remainder (or Difference of the Dividual and Product) be less than the Divisor, (else the Quote-Figure is too little;) and thus we must find the Quote by trials. But to prevent too many useless trials, we have this help, *viz.* Find, by *Case I.* how oft the first Figure on the left is contained in the first on the left of the Dividual, when this and the Divisor have equal number of Places; But in the first two Figures on the left of the Dividual, when this has one place more than the Divisor; and this Number limits the guessing, so that you cannot take a greater; and if this happens to exceed 9, [which it will in no Case but the last, and that where the first of the two Figures in the Dividual is equal to the Divisor-Figure; for it is certain this will be found in these at least 10 times,] your guess begins at 9. But then it will often happen that this Number is too great, and we have no other general Rule, or Help here, but to begin at this Figure and make trials, by multiplying the Divisor; for if the Product does not exceed the Dividual, that is the Figure sought: if it does exceed, take the next lesser Figure, and with it make the like trial; and go on so gradually till you find a Figure whose Product does not exceed the Dividual; for then the Remainder will certainly be less than the Divisor, which is the true Proof of the Quote-Figure's being right. Having thus found the first Figure of the Quote, set it down (on the Right-hand of the Dividend) and write the Product of it by the Divisor under the Dividual, and subtract that out of this; and then before the Remainder (on the right) set the next Figure of the Dividend, (or the Figure on the right of the first Dividual,) and this Number is the second Dividual. Then in the same manner as before, find how oft the Divisor is contained in this Dividual; set the Figure found on the Right-hand of the Quote Figure last found; then multiply the Divisor by it: write down the Product under the Dividual, and subtract as before; then to the Remainder prefix the next Figure of the Dividend, and this is the next Dividual to be divided as before. In this manner proceed till every Figure of the Dividend is employed step by step: And *observe*, that if any Remainder with one Figure of the Dividend prefix'd, makes a Number less than the Divisor, set 0 in the Quote, and prefix also the next Figure; and so on.

All the Quote Figures thus found, taken in order as they are placed as one Number, is the true Quote sought; and the last Remainder is what the Dividend contains over so many times the Divisor as the Quote expresses.

I shall next illustrate this Rule by Examples.

Examp. 1.

Div^r. Div^d. Quote.

4) 6392 (1598.

```

  4
  23
  20
  ---
   39
   36
   ---
    32
    32
    ---
     00
  
```

To divide 6392 by 4, I proceed thus: I seek how often the Divisor 4 is contained in 6, (the first Figure of the Dividend) which is but once; therefore I set 1 in the Quote; then $1 \times 4 = 4$ (or the Divisor multiplied by the Quote is 4,) which I write under the Dividual 6, and subtracting, the Remainder is 2, to which I prefix 3, the next Figure of the Dividend, and then I take 23 for my next Dividual; and examining how oft 4 is contained in it, I find 5 times, (for had I taken 6, it were too great; for $6 \times 4 = 24$, which is greater than the Dividual 23; and had I taken 4, it were too little, for $4 \times 4 = 16$, and $23 - 16$, is 7, which is greater than the Divisor 4;) therefore I place 5 in the Quote on the Right of the former; and

under 23 set $20 = 5 \times 4$; then subtracting, the Remainder is 3, to which I prefix the next Figure of the Dividend, *viz.* 9; then is 39 my next Dividual: and in this I find the Divisor 4 contained 9 times, which I write in the Quote on the right of the former; then $9 \times 4 = 36$, which I write under the Dividual 39, and the Remainder is 3, to which prefixing the next (and last) Figure of the Dividend 2. This Number, *viz.*

32 is my next (and last) Dividual, in which the Divisor is contained 8 times, which I set in the Quote on the right of the preceding Figures; then setting down the Product 32 ($= 8 \times 4$), there is no Remainder; and the Quote sought is 1598, that is, 4 is contained in 6392, 1598 times.

Examp. 2.

36) 85609 (2378.

72
136
108
280
252
289
288

1 Rem^r.

To divide 85609 by 36, I proceed thus: There are two Places in the Divisor, therefore I take the first two Places of the Dividend, viz. 85, which making a Number greater than the Divisor, I take them for the first Dividual, and seeking how oft 3 (the first Figure of the Divisor) is contained in 8 (the first Figure of the Dividual, because they have equal Places) I find it 2 times, which I find also by trial to be the true Quote of 85 by 36; so I place 2 in the Quote, and subscribing the Product 72 ($= 2 \times 36$) I subtract it from 85, the Remainder is 13, to which prefixing the next Figure of the Dividend 6, my next Dividual is 136. Then I seek how oft 3 (the first Figure of the Divisor) is contained in 13, the first

two of the Dividual, (because it has one place more than the Divisor,) and I find it 4 times: but this is too great for the whole Divisor, (because $4 \times 36 = 144$) therefore I try the next Figure 3, and find it right; therefore I set 3 in the Quote, and subscribing the Product 108 ($= 3 \times 36$), I subtract it from 136, and the Remainder is 28; to which prefixing 0, the next Figure of the Dividend, my next Dividual is 280. Then I seek how oft 3 is contained in 28, and I find 9 times; but this is too great, (for $9 \times 36 = 324$.) I take again 8, and find it also too great, (for $8 \times 36 = 288$) and at last I find 7 to be right; therefore I set 7 in the Quote, and subscribing the Product 252 ($= 7 \times 36$) the Remainder is 28; to which prefixing 9, the next and last Figure of the Dividend, I have for my next and last Dividual 289, in which I find as before, that the Divisor is contained 8 times, and 1 remains. So the true Quote is 2378, and 1 over; which is $2378\frac{1}{36}$.

Examp. 3.

465) 2744897 (5903

2325
4198
4185
1397
1395

2 Rem^r.

To divide 2744897 by 465, I proceed thus: my first Dividual is 2744; I seek how oft 4 (the first Figure of the Divisor) is contained in 27 (the first two of the Dividual.) I find it 6; but this is too great, and I take 5, and find it right. Then multiplying and subtracting, and prefixing to the Remainder the next Figure of the Dividend, my next Dividual is 4198; and here I find the Divisor contained 9 times. Then proceeding as before, my next Dividual is 139; which being less than the Divisor, I set 0 in the Quote, and then prefix another Figure; so that my next Dividual is 1397, in which

the Divisor is contained 3 times, and the Remainder is 2; so the true Quote is 5903; or, taking in the Remainder, it is $5903\frac{2}{465}$.

Examp. 4.

462) 3235386 (7003

2234
1386
1386
0000

To divide 3235386 by 462. The first Dividual is 3235, in which the Divisor is contained 7 times, and the Remainder is 1; the next Dividual is 13, in which the Divisor is not contained; therefore I set 0 in the Quote, and bringing down the next Figure, I have 138 for a new Dividual, which is also less than the Divisor, therefore I set another 0 in the Quote; and bringing down another Figure, the next Dividual is 1386, in which the Divisor is contained 3 times, and 0 remains.

Examp. 5.

372) 149100 (400

1488

300 Rem^r.

To divide 149100 by 372. The first Dividual is 1491, and the Quote of this is 4, then the Remainder is 3, and the rest of the Dividuals are 30 and 300, which are each less than the Divisor, and therefore the Quote is 400.

Examp. 6.

$$\begin{array}{r} 62 \overline{) 210800} \quad (3400 \\ \underline{186} \\ 248 \\ \underline{248} \end{array}$$

To Divide 210800 by 62, the Quote is 3400.
 In Cases like this, when there is no Remainder in any step, and that the Remaining Figures of the Dividend are all 0's, we have no more to do than to join as many 0's to the preceding Figure, found in the Quote.

These Examples will be sufficient to make a diligent Learner understand this Rule; and I shall only set down a few more Examples, with their Answers, leaving the Operation for an Exercise to the Student.

Examp. 7. To divide 5713070046 by 678, the Quote is 8426357.

Examp. 8. To divide 606994827 by 8376, the Quote is $724683 \frac{1}{2}$.

Examp. 9. To divide 293682135936 by 8405, the Quote is $34960078 \frac{1}{2}$.

In order to the *Demonstration* of the preceding Rule, and for the sake of some other special Rules to follow, we must premise the following Truths in the Theory of *Division*.

L E M M A I.

1. If two Numbers consist of an equal Number of Places, the lesser is not contained in the greater above 9 times.

2. Again, If the greater of two Numbers has but one Place more than the lesser; and supposing also that, excluding the first Figure on the Right of that greater Number, the remaining Figures on the Left make a Number less than the lesser given Number, then this lesser Number is not contained in the greater above 9 times.

DEMON. Part 1. If a Cypher is prefix'd to the lesser of two Numbers, (which have both the same Number of Places,) it is thereby multiplied by 10; and consequently that is the least Number which contains it 10 times: but the other given Number having one Place fewer than this Product, is a lesser Number, and consequently does not contain the lesser given Number 10 times, or does not contain it above 9 times. *Example,* 11 is less than 99; but is not contained in it 10 times, for 10 times 11 is 110, which is greater than 99. *Universally,* Let A be the greater, and B the lesser of two Numbers having an equal Number of Places; $B \times 10$ contains B precisely 10 times, and it is a Number that has one Place more than B or A, and consequently is a greater Number than A; wherefore B is not contained 10 times in A.

Part 2. The lesser given Number (as 476) is greater, by supposition, than as many Places (475) on the left of the greater given Number (4759;) and must exceed it by at least 1: therefore 10 times this lesser Number, (*viz.* 4750,) must want at least 10 of 10 times the given lesser Number, (*viz.* 4760.) But whatever Digit we add to this deficient Product, or put in the Place of the 0, (making in the present *Examp.* 4759,) it cannot make up the defect of 10; and therefore the given lesser Number (476) is not contained in the given greater Number (4759) 10 times, or not above 9 times.

SCHOL. The second Part of this *Lemma* is but a particular Case (accommodated to our present purpose) of a more general *Theorem*; which is this, *viz.* If any Number A, is greater than another, B; and if B is multiplied by any Number R, then A is not contained R times in R B, nor yet in the Sum of R B, and any Number N which is less than R; *i. e.* in $R B + N$.

The *Reason* is; since A is not once contained in B, neither is R A once contained in R B, which must want at least as many Units as R to make it equal to R A; and since N is less than R, $R B + N$ cannot be equal to R A; *i. e.* A, which is contained precisely R times in R A, is not contained R times in $R B + N$, which is less than R A.

L E M M A II.

If any Number N is resolved into any Parts $A, B, C, \&c.$ i. e. if $N = A + B + C; \&c.$ then,

1. If all these Parts $A, B, C, \&c.$ are severally Multiples of any Number D , or all except one; then dividing $A, B, C, \&c.$ severally by D , the Sum of the Quotes is equal to the Quote of $N \div D$. (*Examp. 1, 2, 3.*) And the Remainder in the Division of that Part which is not a Multiple of D , is the Remainder in the Division of $N \div D$.

2. If there are more than one of the Parts of N , that are not Multiples of D ; and if the Sum of the Remainders, in the Division of these Parts that are not Multiples of D , is less than D , then the Sum of the Integral Quotes, is the Integral Quote of $N \div D$; and the Sum of the Remainders, is the Remainder in the Division of $N \div D$. (*Examp. 4.*)

3. If the Sum of the Remainders is equal to, or greater than D ; then, being divided by D , and the Integral Quote added to the Sum of the Integral Quotes of the Parts of N , this last Sum is the Integral Quote of $N \div D$; and the Remainder in the Division of the Sum of the Remainders, is equal to the Remainder in the Division of $N \div D$. (*Examp. 5.*)

DEMON. The Reason of all these Articles is easily seen from the Equality of the Whole and all its Parts. In the first and second Article it is obvious: for as oft as D is contained in $A, B, C, \&c.$ severally, so oft at least as the Sum of these times, it must be contained in the whole N ; and if the Sum of the Remainders in the Division of the Parts $A, B, C, \&c.$ is less than D , then it is plain that D is contained no oftner in N than the Sum of the times it is contained in all its Parts $A, B, C, \&c.$ and the Remainder in $N \div D$, must be the Sum of the Remainders in the Division of the Parts, when this Sum is less than D ; but if this Sum is equal to, or exceeds D , (as supposed in Article 3.) then since these Remainders are Parts of the Dividend, it is evident, that as oft as D is contained in their Sum, that must be added to the Sum of the times it is contained in $A, B, C, \&c.$ and this last Sum is the times it is contained in N ; and the Remainder on the Division of the Sum of the Remainders, is the Remainder in dividing N by D .

Examp. 1.

$D \ N = A + B + C.$
Divisor 4) $36 = 16 + 12 + 8.$ Dividends.
Quotes $9 = 4 + 3 + 2.$

Examp. 2.

Divisor 5) $48 = 25 + 15 + 8.$ Dividends.
Quotes $9 = 5 + 3 + 1.$
Remainders $3 = 3.$

Examp. 3.

Divisor 6) $46 = 24 + 18 + 4.$ Dividends.
Quotes $7 = 4 + 3 + 0.$
Remainders $4 = 4.$

Examp. 4.

Divisor 6) $53 = 20 + 27 + 6.$ Dividends.
Quotes $8 = 3 + 4 + 1.$
Remainders $5 = 2 + 3.$

SCHOL. If we take the complete Quotes by Fractions made of the Remainders and Divisor; then it is an *Universal Truth*, That the Sum of the Quotes of the Parts of N divided severally by any Number D , is equal to the Quote of N divided by D . For the fractional Parts of the Quotes have all the same Denominator D , and their Numerators are the several Remainders. But from the nature of Fractions it is obvious, that several Fractions having the same Denominator, and being referred to the same Integer, their Sum is the Sum of the Numerators, applied as a Numerator to the same Denominator. *Examp.* $\frac{2}{7}$ of any thing, and $\frac{3}{7}$ of the same thing, make $\frac{5}{7}$. So that if the Sum is an improper Fraction, we find its Value in a whole Number, or with a proper Fraction annex'd, by dividing the Numerator

Examp. 5.

Divisor 6; $88 = 16 + 29 + 13$. Divid.
 Quotes $1 + 7 + 4 + 2$.
 Remainders $4 + 4 + 5 + 1$.

In this last Example, the Sum of the Remainders is $4 + 4 + 5 + 1 = 10$, which is greater than the Divisor 6; and being divided by it, the Quote is $10 \div 6 = 1$, and 4 remains. Then this Quote 1 added to the Sum of the former Quotes, the Sum is $7 + 4 + 2 + 1 = 14$, the Quote of $88 \div 6$, and the Remainder of $88 \div 6$, is the same as that of $10 \div 6$, which is 4.

merator by the Denominator, (as before explained.) It is plain then, that if the Fractions belonging to the complete Quotes of the Parts of N divided by D, are added together, and the Value of the Sum added to the Integral Quotes, the Operation is the same as expressed in the *Lemma*: for it is adding the Remainders; and if their Sum exceeds the Divisor, taking the Number of times the Divisor is contained in it, and adding this Quote to the Sum of the Integral Quotes; which makes the Universal Truth here proposed evident.

COROLLARIES.

I. If a Number N is resolved into any Number of Parts, and these Parts be divided severally by any Number D, in this manner, *viz.* First divide one Part, and if there is a Remainder, add it to another Part, and divide the Sum; and so on, adding the Remainder of every Division to the next Part; and if any Part with the preceding Remainder is less than the Divisor, then we add another. Having thus gone through all the Parts, the Sum of the Quotes is the Quote of the first Number N divided by the same Divisor D; and the last Remainder in the Division of the Parts, is the Remainder in the Division of N by D.

That this is in Effect the same Case as the first Article of the preceding *Lemma*, or a plain Consequence of it, will be obvious by considering, that if the first Remainder is taken out of the first Dividend, it leaves a Multiple of the Divisor, *viz.* the Product of the Divisor and Quote; and the same being true in all the rest of the Steps, it follows that the Case is the same as if N were resolved into Parts equal to these Multiples of the several Quotes by the Divisor; all which Multiples with the last Remainder make up the Dividend N. For Example, $50 = 17 + 8 + 25$, then $17 \div 3 = 5$, and 2 remains, which added to 8, makes 10; then $10 \div 3 = 3$, and 1 remains, which added to 25, makes 26; and $26 \div 3 = 8$, and 2 remains; lastly, the Sum of the Quotes is $5 + 3 + 8 = 16$, the Integral Quote of $50 \div 3$; and the last Remainder 2, is the Remainder of $50 \div 3$. And this Work is the same in effect as if we resolve 50 into these Parts, $15 (= 3 \times 5) + 9 (= 3 \times 3) + 24 (= 3 \times 8)$.

Or this Truth is plain independently of the *Lemma*, because the Divisor is taken out of every Part of the Dividend as oft as possible, by carrying the Remainder of every Part forward to the next.

II. If the same Divisor D is applied to two different Dividends, whereof the greater is a Multiple of the lesser, as N and N m ; then if N contains D, a number of times q , without a Remainder; N m will contain D, m times as oft as N does, or $m q$ times; *i. e.* if $N \div D = q$, then $N m \div D = m q$. Again, if $N \div D$ has a Remainder r , then N m will contain D at least $m q$ times with a Remainder equal to $m r$. And if $m r$ is equal to, or greater than D, then, as oft as D is contained in $m r$, so many times oftner than $m q$ is it contained in N m . The Deduction of this from the *Lemma* is plain; because N m is resolvable into $N + N + N$, &c. taking N as oft as m expresses; so that if $N \div D = q$, and r remains; then D is contained in N m at least $m q$ times, with a Remainder equal to $r m$. See these Examples, wherein $m = 100$.

Examp.

<p>Examp. 1. $\begin{array}{r} D \ N \ q. \\ 3) 18 \end{array} \begin{array}{l} 6. \\ \\ \\ \end{array}$</p> <p style="margin-left: 40px;">$\begin{array}{r} N m \ m q. \\ 3) 1800 \end{array} \begin{array}{l} 600. \\ \\ \\ \end{array}$</p>	<p>Examp. 2. $\begin{array}{r} D \ N \ q. \\ 3) 23 \end{array} \begin{array}{l} 7. \\ 21 = D q. \\ \hline 2 = r. \end{array}$</p>	<p>$\begin{array}{r} D \ N m \ m q. \\ 3) 2300 \end{array} \begin{array}{l} 700. \\ 2100 = D \times m q. \\ \hline 200 = r m. \end{array}$</p>
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SCHOL. These Examples are of the Kind which we have particular Use for in Demonstrating the *Rule of Division*: And we have this further to be carefully remarked in all Examples like the second, viz. That tho' the Remainder (mr) in the second Part of the Example, is greater than the Divisor, whereby the Integral Quote is not so great as the number of times that D may be got in Nm , yet of the Number it wants to be added to it, (which is the Quote of mr by D ;) all the Figures will fall in the Places of the o's standing on the right of that Part of the Quote which is already found, and can never be of the same local Value with any of the other Figures. So 3 is found in 2300 as many times oftner than 700 times, as the Quote of 200 by 3; yet no Figure of this Quote can rise to the Place of 100; the Reason of which, and of all such Cases, is explained in *Schol. to Lemma I.* For 3 being greater than 2, is not contained 100 times in 200. And which will also be true, tho' we set any other Figure in the Place of the o's that stand on the right of the Remainder, since the Remainder without these o's is less than the Divisor. See the *Scholium* referred to.

III. If $N (= A + B)$ is a Multiple of C ; and if A is also a Multiple of C , then so must B be. Again, if $N = A + B + C + D$, &c. and if N is a Multiple of R ; and also if each of the Parts of N to the last, are Multiples of R , so must that last be.

L E M M A III.

If one Number is divided by another, and the Quote again divided by the same, or any other, and every succeeding Quote again divided as long as you please or can; the last Quote will be equal to the Quote of the first Dividend by the continual Product of all these Divisors.

DEMON. I. We shall first suppose the several Dividends are Multiples of the Divisors; and in this Case it will easily appear, as in the annex'd Example.

The Reason is this. If we take the last Quote and all the Divisors in a reverse Order, and multiply them continually, they must produce the first Dividend, (by what is already shewn of the mutual Proof of *Multiplication and Division*.) Thus, $4 \times 7 = 28$. $28 \times 3 = 84$. $84 \times 2 = 168$. But we may take these Factors in any Order, they will produce at last the same Number, (by *Lemma II. in Multiplication*.) And if we order them so as the last Quote in Division be the last Factor in multiplying, the Truth proposed will be manifest, thus; Because $4 \times 7 \times 3 \times 2 = 168$. Therefore also $2 \times 3 \times 7 \times 4 = 168$. But $2 \times 3 \times 7 = 42$; therefore $42 \times 4 = 168$; and $168 \div 42 = 4$. The same Reasoning will hold in all Cases, which we may represent by Universal Characters. Thus; if $A \div b = M$; and $M \div c = N$; and $N \div d = q$: then $A \div bcd = q$. For, $qd = N$; $Nc = M$; and $Mb = A$; that is, $qdc b = A$; or $bcdq = A$: But $bcdq \div bcd = q$; i.e. $A \div bcd = q$.

2. Suppose there is a Remainder in each Division, yet the last Quote will still be equal to the Quote of the first Dividend by the Product of all the single Divisors, tho' the Remainder will not be the same as the Remainder of the last Division. That we may see the Truth proposed in this Case, and also how to find by the several Divisors and Remainders

ders, what the Remainder would be upon dividing by the Product of the Divisors, we shall consider the *Example* annex'd; wherein the thing proposed is proved. But to see the Reason of it, we must take the *reverse Multiplication*, as in the Margin on the Right.

$$\begin{array}{r} 3 \overline{) 479} \text{ (159} \\ \text{rem. 2} \end{array}$$

$$\begin{array}{r} 5 \overline{) 159} \text{ (31} \\ \text{rem. 4} \end{array}$$

$$\begin{array}{r} 4 \overline{) 31} \text{ (7} \\ \text{rem. 3} \end{array}$$

then because

$$3 \times 5 \times 4 = 60$$

therefore

$$\begin{array}{r} 60 \overline{) 479} \text{ (7} \\ \text{rem. 59} \end{array}$$

Thus; Take the last Quote and the several Divisors, as so many simple *Factors*, and multiply them continually, taking in the correspondent Remainders with the Product, (to make up the several Dividends.) Now the first thing to be shewn from this Multiplication is the reason why, That tho' the last Product (479) exceeds the continual Product of the *Factors* $7 \times 4 \times 5 \times 3 = 420$, (as it must do, because of the Numbers taken in;) yet it cannot exceed it by a Number as great

$$\begin{array}{r} \text{last quote } 7 : 3, \text{ last rem.} \\ \text{last div. } 4 \end{array}$$

$$\begin{array}{r} 2^{\text{d}} \text{ quote } 31 : 4, 2^{\text{d}} \text{ rem.} \\ 2^{\text{d}} \text{ div. } 5 \end{array}$$

$$\begin{array}{r} 1^{\text{st}} \text{ quote } 159 : 2, 1^{\text{st}} \text{ rem.} \\ 1^{\text{st}} \text{ div. } 3 \end{array}$$

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as the continual Product of the several *Factors* excluding the first 7, (*i. e.* the several Divisors, *viz.* $4 \times 5 \times 3 = 60$;) which is thus shewn.

The Number by which the last Product, or Sum (479) exceeds the continual Product of all the simple *Factors* (420,) is plainly to be found thus; *viz.* Take the Product of the last Remainder (3) by the last Divisor but one, (5;) then to this Product (15) add the next preceding Remainder (4,) and multiply the Sum (19) by the next preceding Divisor (3,) and to this Product (57) add the next preceding Remainder (2,) and so on: for the last Remainder 3 is taken in with the first Multiplication; then it is multiply'd in the second Multiplication by 5, with which the second Remainder is taken in, (which makes $15 + 4 = 19$.) Then is all this multiply'd in the third Multiplication by 3, and the first Remainder 2 is taken in, and the whole is 59; so the true Remainder sought is 59. But this must be always less than the Product of the Divisors, because in the several Multiplications the Numbers taken in are less than the correspondent Multipliers, (for they are Remainders of a Division wherein that Multiplier was the Divisor;) wherefore the Product of any of the Divisors by the following Remainder, (or Remainder of the next Division,) with the present Remainder added, is less than the Product of the same Divisor by the following Divisor, and consequently the continual Product of all the Divisors is greater than the Product of all the Divisors (excluding the last,) with the Remainders taken as directed. This being once clear, the Truth concerning the last Quote is manifest; for nothing else can make it different from the last Case but these Remainders, or Numbers added in the Multiplication, which can never make the Quote less: and, by what is now shewn, they cannot make it greater, because all the Increase upon the last Dividend is less than the total Divisor, (or Product of the particular Divisors.)

As to the fractional Part of the Quote, it will be of the same Value, tho' not of the same Expression. But the Truth of this will not appear so easily till we have learnt the *Doctrine of Fractions*, where you'll find it particularly explained.

DEMON-

DEMONSTRATION of the preceding Rule of CASE II. of DIVISION.

As to finding every one of the Figures of the Quote singly, as the true Quotes of the Divisor into the several Dividuals considered by themselves, we need no further Demonstration; because each of them is found by trial, and proved to be true by a certain and infallible Mark, (discovered in *Corol. 2. to the Definition*) before it is admitted. All that I have more to add, is to observe, That you have in *Lemma 1.* the Reason that the Divisor can never be found in the Dividual, (or Partial Dividend, as it is to be taken by the Rule) above 9 times.

The only thing then that remains to be proved is, That the several Figures of the Quote taken as one Number, according to the Order in which they are placed, is the true Quote of the total Dividend by the Divisor; and this will be easily shewn, thus: It is plain, that in this Operation we have resolved the Dividend into Parts, and divided them severally in the manner proposed in *Corol. 1. Lem. 2.* For we have first taken one Dividual, then added the Remainder of that to another Part of the Dividend, and after dividing this, we have added to the Remainder another Part of the Dividend, and so on. But we have considered the several Dividuals without regard to the Places they possess in the Whole Dividend, and thereby taken them in a less than their true Value; but if that Defect is supplied by placing the several Quotes, (or Parts of the total Quote) so as they have the true Value they ought to have from the complete Value of their respective Dividuals, (or Parts of the Dividend,) and that by so placing them they make one Number equal to the Sum of their complete Values; then is that Number the true Quote sought, (by the said *Corol. 1. Lem. 2.*) But thus it actually is; for the complete Value of the first Dividual is 10, or 100, &c. times the Value in which it is taken in the Operation, according as there is one, two, &c. Figures standing before it: Also its quote Figure standing first on the Left-hand, there are as many Figures of other Quotes set before it, as the Number of remaining Figures in the Dividend; because for each of these there is, by the Rule, a Figure put in the Quote: therefore this first quote Figure receives by its Place, a Value 10, or 100, or 1000, &c. times its simple Value, according as there are one, or two, or three, &c. Figures before it, corresponding to the true Value of the Dividuals, (as ought to be done by *Corol. 2. Lem. 2.*) Therefore this first quote Figure taken in its complete Value from the place it stands in, is the true Quote of the Divisor in the complete Value of the first Dividual. For the same reason, all the rest of the Figures in the Quote taken according to their Places, are each the true Quote of the Divisor in the complete Value of their Dividuals; because as the first Figure on the right of each succeeding Dividual, is one Place more to the right of the preceding, (or has one Figure fewer standing before it,) so ought their Quotes to have; and so are they also ordered: Consequently taking all the quote Figures in order as they are placed by the Rule, they make one Number, which is equal to the Sum of the true Quotes of the several Parts of the Dividend; which is therefore the true Quote of that whole Dividend.

To leave no Obscurity in this Demonstration, I shall illustrate it by two Examples. In which I shall set down the Dividuals, Quotes, and Remainders according to their true Values.

Examp.

Examp. 1.

Divisor. Dividend.

$$\begin{array}{r}
 36 \overline{) 85609} \quad \text{Quotes.} \\
 \text{1st dividual} \quad 85000 \quad (\quad 2000 \\
 36 \times 2000 = 72000 \\
 \text{1st remain.} \quad 13000 \\
 \quad \text{add} \quad 600 \\
 \text{2d dividual} \quad 13600 \quad (\quad 300 \\
 36 \times 300 = 10800 \\
 \text{2d remain.} \quad 2800 \\
 \quad \text{add} \quad 00 \\
 \text{3d dividual} \quad 2800 \quad (\quad 70 \\
 36 \times 70 = 2520 \\
 \text{3d remain.} \quad 280 \\
 \quad \text{add} \quad 9 \\
 \text{4th dividual} \quad 289 \quad (\quad 8 \\
 36 \times 8 = 288 \quad 2378 \text{ Sum} \\
 \text{last remain.} \quad 1 \quad \text{of the Quotes.}
 \end{array}$$

In the first Example, the Dividend 85609 is resolved into these Parts, *viz.* 8500 + 600 + 00 + 9. For tho' the first Dividual is considered as 85, yet it is truly 85000; and therefore its Quote instead of 2, is 2000, and the Remainder 13000; and so of the rest, as you see in the Operation. But if we take the Multiples of the Divisor by the several Quotes, with the last Remainder, and consider the Dividend as distributed into these Parts, (which are here 72000 + 10800 + 2520 + 288 + 1,) then the Work is reduced to the Conditions of *Lemma 2. Article 2.*

Examp. 2.

Divisor. Dividend.

$$\begin{array}{r}
 465 \overline{) 2744897} \quad \text{Quotes.} \\
 \text{1st dividual} \quad 2744000 \quad (\quad 5000 \\
 465 \times 5000 = 2325000 \\
 \text{1st rem.} \quad 419000 \\
 \quad \text{add} \quad 800 \\
 \text{2d dividual} \quad 419800 \quad (\quad 900 \\
 465 \times 900 = 418500 \\
 \text{2d rem.} \quad 1300 \\
 \quad \text{add} \quad 97 \\
 \text{3d dividual} \quad 1397 \quad (\quad 3 \\
 465 \times 7 = 1395 \quad 5903 \\
 \text{last rem.} \quad 2
 \end{array}$$

In this second Example, when we have got the second Quote, the Remainder is 1300; then we add the two next Figures of the Dividend, because the Figure of the Quote must be of the same local Value as the last of these Figures: For since 465 is not contained in 139, it is not contained 10 times in 1390; and so the next Figure in the Quote after 9 must be 0, and the significant Figure of the Quote of 1390 ÷ 465, must be in the second Place after 9, *i. e.* in the present Example, in the Place of Units: and therefore we take in also the Figure 7, which is in the Place of Units, to find at once all that Part of the Quote which belongs to the Place of Units; for had we divided 13900 by 465, the Quote is 2, and the Remainder 460; to which adding the last Figure of the Dividend 7, the Sum is 467, in which the Divisor is contained once, and 2 remains; and so

these two Quotes both in the place of Units, *viz.* 2 + 1 make 3, which is more conveniently found, as in the Operation in the Margin. The like reason you'll find in all Cases where there are 0's in the Quote. And for the last Example, take 113764 ÷ 28 = 4063, the first Dividual is 113000, the Quote 4000, and the Remainder 1000; to which if we add 700, the Sum 1700 does not contain the Divisor 28 such a number of times as can fall in the Place of Hundreds; therefore we take in another Figure, which makes 1760; and the Quote 6 falls in the second Place after the preceding Quote 4. The last Figure of the Quote is 3.

§. II. PARTICULAR RULES for contracting the Work of Division in certain Cases: And, for managing it with more Certainty, tho' with more Work, in all Cases.

CASE 1. When the Divisor is a Digit, the multiplying of the Divisor and Quotes, and also the Subtraction of the Products from the Dividuals, may be easily performed without writing down any thing but the Remainders; and these, with the Quotes, set more conveniently, as in this *Example*; wherein 37546 is divided by 4. Thus, 4 in 37 is 9 times, and 1 remains; the Quote 9 I set under the Dividend, and the remainder 1 above; then this Remainder, with the next Figure of the Dividend prefix'd, makes 15; the next Dividual, in which 4 is contained 3 times, and 3 remains; the Quote 3 is set after the preceding Quote, and the Remainder 3 over the 5 of the Dividend. Then is the next Dividual 34, whose Quote is 8, and 2 remains; then the last Dividual is 26, and the corresponding Quote is 6, and 2 remains. *Again*, in this Case it will be very easy to do the Work without writing down the Remainders, only conceiving them, in the places where they ought to be. And the Conveniency of doing it this way, you'll see in Case 3. Observe also, that you may easily use the same Practice, if the Divisor is 11 or 12.

CASE 2. If the Divisor has o's in the first Places next the Right-hand, take no notice of them in the Operation, making the Divisor only the remaining Figures on the Left; and exclude as many Figures, whatever they are, from the Right of the Dividend, as those o's of the Divisor; making the remaining Figures on the Left the Dividend. Having finished this Division, you have found the integral Quote sought: And for the fractional Part, to the Remainder of the Division prefix all the Figures excluded from the Dividend; that is the true Remainder that would happen if the Division were done by the common Method: This Remainder, with the given Divisor, makes the Fraction. But observe, that if there are any o's standing clear on the Right-hand of this Remainder, you may omit them all (in making your Fraction) if they do not exceed the Number of o's excluded from the Divisor; or as many of them, as are equal in Number to those in the Divisor; and omitting the same Number of o's in the Divisor, of the remaining Figures on the Left make your Fraction.

Example 1. To divide 84700 by 4600, I divide 847 by 46, the Quote is 18, and 19 remains; but the true Remainder is 1900: Making this Fraction $\frac{1900}{4600}$, which is equivalent to this $\frac{19}{46}$.

Ex. 2. To divide 3640 by 800, I divide 364 by 8, the Quote is 45, and 4 remains; but the true Remainder is 420: Making this Fraction $\frac{420}{800}$, equal to this $\frac{21}{40}$.

Examp. 3. To divide 68704 by 2400, I divide 687 by 24; the Quote is 28, and 15 remains; but the true Remainder in the Question is, 1504, and the Fraction is $\frac{1504}{2400}$.

Examp. 4. To divide 367854 by 800, I divide 3678 by 8; the Quote is 459, and the Remainder 6; but the true Remainder is 654, and the Fraction $\frac{654}{800}$.

The Reason of this Rule is contained in Lem. 4. Thus, if an equal Number of o's are excluded from Divisor and Dividend, the remaining Figures on the left express like aliquot Parts of them, viz, a tenth Part if one o; a hundredth, if two o's were excluded, and so on. But like aliquot Parts of two Numbers make the same Quote as their Wholes, (by Lem. 4.) *Ex.* 46 and 847 are the hundredth Parts of 4600 and 84700, and so have the same Quote 18, with a Remainder 19, which is the hundredth Part of the Remainder in dividing 84700 by 4600; and therefore the two o's cut off from the Dividend are to be prefix'd to it, to give it the true Value. *Again*, if the Figures excluded the Dividend are not all o's, yet if we suppose them so, the Quote is right: And that it cannot be increased by the Value of the Figures cut off, whatever they are, is plain; because

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cause being prefix'd to the Remainder found by the Rule, they must make a less Number than the given Divisor; since that Remainder is less than the Divisor without its o's, which are as many in number as these Figures prefix'd to the Remainder, which therefore can never make up the Defect. So in *Ex. 4.* when 3678 is divided by 8, the Remainder is 6, which being less than 8, no Figures prefix'd to it can make a Number equal to 8, with as many o's prefix'd. But these Figures being a part of the Dividend, belong to the Remainder; which, instead of 600, as it would have been had the Figures cut off been o's, is 654.

Tho' this is the proper Demonstration of this Rule, yet you may be sufficiently satisfied of the Justness of it, by considering any Example wrought at length; wherein it will easily appear, that by excluding the o's in the Divisor, and as many Figures in the Dividend, we only save the Trouble of writing many superfluous Figures, and yet bring out the same Quote. As in these *Examples.*

$$\begin{array}{r} 4600)84-00(18 \\ \underline{4600} \\ 38-00 \\ \underline{36800} \\ 1900 \end{array}$$

$$\begin{array}{r} 800)367854(459 \\ \underline{3200} \\ 4785 \\ \underline{4000} \\ 7854 \\ \underline{7200} \\ 654 \end{array}$$

As to that part of the Rule for contracting the Fraction, you'll find the Reason of it explain'd in Book II.

CASE 3. If the Divisor is the Product of two or more Digits, and that you can easily discover these Digits; then divide first by any one of these, and the Quote by any other, and so on: the last Quote is that sought. And for the Fraction, multiply the Divisor by the last Quote, and take the Product from the Dividend, you have the Remainder, which would happen by dividing after the common way. Or find it thus; Multiply the last Remainder (of the Work) by the preceding Divisor (or the last but one) and to the Product add the preceding Remainder; this Sum multiply by the next preceding Divisor, and to the Product add the next preceding Remainder, and so on, till you have gone thro' all the Divisors and Remainders to the first. But when there are no Remainders in any of the particular Divisions, the Dividend is a Multiple of the Divisor.

Ex. 1. To divide 9048 by 24, I divide by 4 and 6, because $4 \times 6 = 24$. Thus, $9048 \div 4 = 2262$, then $2262 \div 6 = 377$.

Ex. 2. To divide 754683 by 42, I divide by 6 and 7, because $6 \times 7 = 42$, as in the Margin; wherein the first Quote is 125780, and 3 remains, which I have set over a Line after the Quote; then the second Quote is 17968, and 4 remains, which multiplied into the preceding Divisor 6, produces 24, to which the first Remainder 3 being added, makes 27; so that the true Remainder is 27, and the fractional part of the Quote $\frac{27}{42}$.

Ex. 3. To divide 18472 by 32, I divide by 4 and 8, because $4 \times 8 = 32$; the first Quote is 4618, and nothing remains; the second Quote is 577, and 2 remains, which multiplied into 4, produces 8, the true Remainder. But in this Case, where there is no Remainder in the first Step, the Fraction may be made of the Remainder, and Divisor of the second Step. So here it may be $\frac{2}{8}$.

Ex. 4. To divide 48767 by 15, I divide by 3 and 5, because $3 \times 5 = 15$: the first Quote is 16255, and 2 remains; the second Quote is 3251, and nothing remains; wherefore there is no Product to be added to the first Remainder: and so that is the true Remainder, and the Fraction is $\frac{2}{15}$.

Ex.

Ex. 5. To divide 3428689 by 126, I divide by 3, 6, 7, because $3 \times 6 \times 7 = 126$: the first Quote is 1142896, and the Remainder 1; the second Quote is 190482, and the Remainder 4; the last Quote is 27211, and the Remainder 5, which multiplied into the preceding Divisor 6, produces 30; to which the preceding Remainder 4 added, makes 34; which multiplied by the preceding and first Divisor 3, produces 102; and the Remainder 1 added, makes 103 the true Remainder, and the Fraction is $\frac{103}{126}$.

The Reason of this Rule is explained in Lem 3. and as to the Contraction of the Fraction in Ex. 2. you'll learn the reason of it in Book II.

This Practice is of very good use, especially where the Divisor is the Product of two Digits; because when it is so, they are easily discovered: and the use of it you'll find more remarkably in the next Chap. §. 5. Observe also, that if the Factors of the Divisor are 11 or 12, it's easy to divide by them as by a Digit. Thus to divide by 144, chuse 12, 12, because $12 \times 12 = 144$. For 33 take 3, 11. For 84, take 7, 12.

CASE 4. One who is tolerably acquainted with the Practice of Division, according to the preceding general Rule, may contract the Work, by omitting to write down the Product of every Figure of the Quote by the Divisor; doing it in mind, and gradually as the Product is made, subtracting it from the corresponding Figures of the Dividual; setting the Remainder either above or below the Dividend, in the manner of the following Examples: For it's no matter whether the Figures of any Remainder or Dividual stand all in one Line, if they are duly situated with respect to one another, from Left to Right-hand. Also, instead of setting the Quote on the Right-hand of the Dividend, it may stand as conveniently under or over the Dividend.

Ex. 1. $72849 \div 46 = 1583$, and 31 remains.

That you may perceive the manner of working without Confusion, I shall represent it as it appears separately at every Step.

$\begin{array}{r} 26 \\ 46 \overline{) 72849} \\ \underline{1} \end{array}$	$\begin{array}{r} 3 \\ 268 \\ 46 \overline{) 72849} \\ \underline{15} \end{array}$	$\begin{array}{r} 31 \\ 2686 \\ 46 \overline{) 72849} \\ \underline{158} \end{array}$	$\begin{array}{r} 313 \\ 26861 \\ 46 \overline{) 72849} \\ \underline{1583 \frac{31}{46}} \text{ Quote.} \end{array}$
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The first Dividual is 72, the Quote 1, and Remainder 26. The second Dividual is 268, the Quote 5, and Remainder 38; which is found gradually; Thus, $5 \times 6 = 30$; then 8 (the first Figure of the Product) from 8 (the first Figure of the Dividual) remains 8. Again $5 \times 4 = 20$, and 3 (the Number of 10's carried from the last Product) is 23: then 23 from 26 (of the Dividual) remains 3; whence the next Dividual is 384; the Quote 8, and Remainder 16; Found thus, $8 \times 6 = 48$, then 8 (of the Product) from 4 (of the Dividual) cannot be taken, but from 14, and 6 remains: Again $8 \times 4 = 32$, and 4 (from the last Product) is 36; then 36 from 37 (instead of 38 of the Dividual, because 1 was taken from the 8 in the last Step to make 14) leaves 1. Or it comes to the same thing, if to the Product we add 1, for the 10 that was borrowed in the last step; so the Product 36 and 1 (borrowed) is 37; which taken from 38, there remains 1. The next Dividual is 169, the Quote 3, and Remainder 31.

By this Example you may understand how to do, or examine others. See the following.

<p>Ex 2. $\begin{array}{r} 33 \\ 68 \overline{) 24786} \\ \underline{364 \frac{34}{68}} \text{ Quote.} \end{array}$</p>	<p>Or it may stand thus. $\begin{array}{r} 3 \\ 304 \\ 68 \overline{) 24786} \\ \underline{364 \frac{34}{68}} \end{array}$</p>	<p>Ex. 3. $\begin{array}{r} 23 \\ 2527 \\ 445545 \\ 467 \overline{) 247893} \\ \underline{6954 \frac{375}{467}} \text{ Quote.} \end{array}$</p>
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Observe, When in any Step there is 10 borrowed in the Subtraction, then the Remainder will be always greater than the Subtrahend, (because if any Digit is taken from the Sum of 10 and a lesser Digit, the Remainder must be greater than that lesser Digit.) And when you come to the next Step, look back upon the last Remainder, its being greater than the Subtractor over which it stands, is a certain sign that 10 was borrowed in the last Step, and consequently that 1 for that 10 is to be taken from the Subtractor, or added to the Product in this Step: And this Observation is very useful, because we are aptest to forget this 1. So in *Ex. 3.* the last Dividual is 2243, the Quote 4, and Remainder 375: Thus found, $4 \times 7 = 28$; then 8 from 13 leaves 5. Again, $4 \times 6 = 24$, and 2 (carried from the last Product) is 26, and 1 (borrowed, because 5 is greater than 3 over which it stands) makes 27; then 7 from 14 leaves 7. Again $4 \times 4 = 16$, and 2 (carried from the last Product) and 1 (borrowed) makes 19, which taken from 22, leaves 3.

In the common way of practising this Method, they dash a Line thro' every Figure of the Dividual, gradually as the Figures of the Remainder are set over them, in order to prevent Confusion; because the next Dividual appears the more distinctly from the Figures of the preceding Dividuals that are thus cancelled. As in this *Example*, represented in all its Steps separately.

$$\begin{array}{r} 9 \qquad 2 \qquad 21 \\ 24)5726 \qquad 24)5726 \qquad 24)5726 \\ 2 \qquad 23 \qquad 238\frac{1}{4} \end{array}$$

Observe, As it is to the same Purpose whether we write the Remainders over or under the Dividend, so the Method of placing them has a little Confusion in it, which is helped by dashing the Figures. But I think it a more convenient

way to set the Remainders under the Dividuals, and so as the Figures of the same Remainder be in a Line; leaving the next Figure of the Dividend, which makes up the Dividual, where it stands, and setting the Quote over the Dividend, as in these *Examples*.

$$\begin{array}{r} 238 \text{ Quote.} \\ 24)5726 \text{ Dividend.} \\ 9 \text{ } \\ 20 \text{ } \\ 14 \text{ } \end{array} \left. \vphantom{\begin{array}{r} 238 \\ 24)5726 \\ 9 \\ 20 \\ 14 \end{array}} \right\} \text{Remainders.}$$

I shall recommend this Method of Division only to such as, by Practice, have acquired a Habit of close and careful Attention; otherwise 'tis too difficult. But there are some particular Circumstances wherein 'tis very easy and convenient, as in the two following *Cases*.

$$\begin{array}{r} 364 \text{ Quote.} \\ 68)24786 \text{ Dividend.} \\ 43 \text{ } \\ 30 \text{ } \\ 34 \text{ } \end{array} \left. \vphantom{\begin{array}{r} 364 \\ 68)24786 \\ 43 \\ 30 \\ 34 \end{array}} \right\} \text{Remainders.}$$

CASE 5. If the Divisor has the Figure 9 in all its Places, as 9, 99, 999, &c. then the Quote is either 1, when the Dividual is equal to the Divisor; or if the Dividual has one place more than the Divisor, the Quote is either the first Figure on the Left of the Dividual, or the next greater Figure. Thus, if the first Figure added to the remaining Figures (taken as one Number) makes a Sum less than the Divisor, that first Figure is the Quote, and that Sum is the Remainder, after the Product of the Divisor and Quote is taken out of the Dividual. But if that Sum is equal to, or greater than the Divisor, the Difference is the Remainder, and the Quote is the Figure next greater than the first Figure of the Dividual.

Thus the Quote and Remainder being easily found, (without the Product) we may chuse the Method of the preceding *Case* for placing them; and we may also make it shorter, by not writing down such Figures of the Remainder which are the same with their Correspondents in the Dividual, but leaving them where they stand.

Ex.

Ex. 1.
$$\begin{array}{r} 35 \\ 72 \overline{) 468793} \\ \text{Quote. } 4735 \frac{28}{99} \end{array}$$

Here the first Dividual is 468, the Quote 4, and the Remainder 72 ($=68 + 4$.) The next Dividual is 727, the Quote 7, and Remainder 34 ($=27 + 7$.) The next Dividual is 349, the Quote 3, and Remainder 52 ($=49 + 3$.) The next Dividual is 523, the Quote 5, and Remainder 28.

Ex. 2.
$$\begin{array}{r} 0 \\ 92 \quad 4 \\ 999 \overline{) 67934582} \\ \text{Quote. } 68002 \frac{584}{999} \end{array}$$

Here the first Dividual is 6793, the Quote 6, and Remainder 799 ($=793 + 6$.) The next Dividual is 7994, the Quote is 8 ($=7 + 1$), and the Remainder 2, (because $994 + 7 = 999 + 2$.) The next Dividual 25, and the next again 258, being each less than the Divisor, the Quote Figures are 0; the last

Dividual is 2582, the Quote 2, and Remainder 584 ($=582 + 2$.)

By these Examples you may easily do or examine any other; see the following.

DEMONSTR. The thing to be demonstrated in this *Rule*, is the way of finding the Quote and Remainder when the Dividual has one place more than the Divisor, which, upon a little Attention, will be very obvious. For 99 wants 1 of 100, (and any Number expressed thus by 9's, wants 1 of a Number expressed by 1, and as many 0's.) And however oft 100 is contained in any Number of the same Number of places, (which is always as oft as the first Figure on the Left expresses, the remaining Figures on the Right being the Remainder; so $462 \div 100 = 4$, and 62 remains) so oft at least must 99 be contained in it: and the Remainder will be the Sum of the Remainder when divided by 100, and of the Quote or first Figure; which is the Remainder when 99 is taken out of any Number of Hundreds. Thus, $468 \div 99 = 4$, and 72 remains, *viz.* $68 + 4$; for in $400 \div 99$, the Remainder is 4. And therefore, since $4 + 68$ is less than 99, it must be the true Remainder in dividing 468 by 99. Again, $697 \div 99 = 7$ and 4 remains; for 99 is contained in 600 six times, and 6 remains; then $6 + 97 = 99 + 4$, in which 99 is contained once, and 4 remains: Wherefore 99 is contained in 697, 7 times, and 4 remains. The same Reasoning holds in all Cases.

Examp. 3.
$$\begin{array}{r} 73 \\ 7622 \\ 999 \overline{) 746985} \\ \text{Quote. } 747 \frac{212}{999} \end{array}$$

Examp. 4.
$$\begin{array}{r} 3 \quad 9 \\ 99 \overline{) 39976} \\ \text{Quote. } 403 \frac{79}{99} \end{array}$$

Examp. 5.
$$\begin{array}{r} 92 \quad 4 \\ 999 \overline{) 67934582} \\ \text{Quote. } 68002 \frac{584}{999} \end{array}$$

Examp. 6.
$$\begin{array}{r} 71 \\ 99 \overline{) 99467} \\ \text{Quote. } 1004 \frac{71}{99} \end{array}$$

Examp. 7.
$$\begin{array}{r} 9 \quad 61 \\ 99 \overline{) 6039457} \\ \text{Quote. } 61004 \frac{61}{99} \end{array}$$

Examp. 8.
$$\begin{array}{r} 9 \\ 99 \overline{) 40596} \\ \text{Quote. } 410 \frac{9}{99} \end{array}$$

Examp. 9.
$$\begin{array}{r} 809 \\ 99 \overline{) 67419} \\ \text{Quote. } 681 \end{array}$$

The Use and Conveniency of this *Case*, you'll find chiefly in *Chap. 4. Book 5.* where such Divisors frequently occur.

CASE 6. If the Divisor has any other Figure than 9 in all its Places, as 44, 666, &c. doing the Work in the manner of *Case 4*, will be easier than if the Figures were all different: But it will be still easier with a little more Work. Thus, divide by 11 or 111, &c. with as many Places as the Divisor has; and here the Quote is easily seen, for it is either the first Figure on the Left of the Dividual, or the next lesser Number; or 0, if the Dividual has the same Number of Places as the Divisor. But if the Dividual has one Place more than the Divisor, the Quote is 9. Then the Product of the Divisor and Quote having the same Quote Figure in all its Places, the Remainder is easily found without writing down the Product, and so may be disposed in the manner of *Case 4*. Then, lastly, divide this Quote by that Digit of which the Divisor consists, by *Case 1*.

Ex.

Examp. 1. $44)562387$.

Thus,

$$\begin{array}{r} 11261 \\ 11)562387 \\ 4)51126\frac{2}{4} \\ 12781\frac{2}{4} \end{array}$$

Examp. 2. $777)1672869$.

Thus,

$$\begin{array}{r} 110 \\ 230872 \\ 111)1672869 \\ 4)42097\frac{10}{7} \\ 10524\frac{21}{77} \end{array}$$

A GENERAL METHOD to make DIVISION certain and easy.

The Work of *Division* may be made more certain and easy with the writing a few more Figures; by making a Table of the Products of the Divisor by all the Digits; as was done in §. 2. of the preceding *Chap.* for Multiplication. To be used thus; Seek the Dividual, or the next lesser Number, in the Table against it is the Quote Figure, and that Number itself is the Product of the Divisor and Quote: By this means the Work may be done as fast as Figures can be written. Nor need the Products be copied out of the Table, but taken where they stand. The Remainders may be found and written under the Dividual: And here also we may chuse to write the Dividuals and Remainders under the Dividend, and without bringing down the Figures of it to the Remainders, taking them where they stand, as has been shewn in *Case 4*.

Example.

1	483)35268749(725604 $\frac{2}{3}$
2	986	3401
3	1469	1258
4	1952	986
5	2435	2727
6	2918	2435
7	3401	2924
8	3884	2918
9	4367	69

Here the first Dividual is 3526; the nearest Number to it in the Table is 3401, against 7, which is therefore the Quote, and 3401 is the Product. The next Dividual is 1258, whose next Number in the Table is 986, &c.

But the Work may be made without writing the Products out of the Table; and it will stand thus.

1	483)35268749(725604 $\frac{2}{3}$
2	986	125
3	1469	272
4	1952	292
5	2435	6
6	2918	
7	3401	
8	3884	
9	4367	

Still be more easy, if we write the Table upon a separate bit of Paper, so as we can place each Product under the Dividual.

And, lastly, in such Questions and Calculations where the same Number must be frequently used as a Divisor, this Method of a Table, especially a moveable one, is most convenient, as you'll find afterwards.

NEPER'S RODS may also be used for making the Table of the Divisor.

Of the PROOFS of DIVISION.

1. We have already explained one *Proof* of *Division* by *Multiplication*, which is this; viz. The Product of the Divisor and Integral Quote added to the Remainder, is equal to the Dividend.

2. But it may be also proved by *Division*. For the *Dividend* being divided by the Integral Quote, the *Quote* of this *Division* will be equal to the former *Divisor*, with the same *Remainder*. Thus, 3 is contained 4 times in 14, and 2 remains: But 4 times 3 = 12; therefore 4 must be contained 3 times in 14, with the same Remainder 2; as it actually is. The same Reason is good in all Cases.

3. Lastly, *Division* may be proved by casting out the 9's. Thus; Subtract the *Remainder* out of the *Dividend*, what remains here ought to be the Product of the *Divisor* and *Quote*; which you may prove by casting out the 9's, as was done in *Multiplication*.

C H A P. VII.

Of APPLICATE NUMBERS.

Explaining the preceding Fundamental Operations, as they are Applicable to Questions about Particular Things, with their Circumstances in Human Affairs.

§. 1. Of Applicate Numbers; and their Distinction of Simple and Mix'd.

With TABLES of the Variety of COINS, WEIGHTS, and MEASURES of GREAT BRITAIN.

WHY Numbers are called *Applicate*, we have learned already. But now we are to consider, That for the Use of Society, it was necessary that certain greater Quantities should be subdivided into other lesser ones; and these again into others lesser; each having its distinct and proper Denomination; but all considered as subordinate Species of the greater; in order to the giving and receiving more or less of any Goods or Commodity, as occasion should require. As for Example, One Pound (of Money) is divided into 20 Shillings, one Shilling into 12 Pence, and one Penny into 4 Farthings. These several lesser Quantities, as they have distinct Denominations, are as really *Integers* of their own kind as the greater, of which they are a Part; and a Number of each Species considered by itself, is called a *simple Number*; as 48 Pounds, or 56 Shillings, &c. But when we take together a Number of several Species, taking still less of each inferiour Species than what makes an *Unit* of the higher, and considering it as a Part of that *Unit*, this makes a *mix'd Number*. For Example; 48*l.* 14*s.* 9*d.* 2*f.* is one mix'd Number. Again consider, that as the Numbers of the inferiour Species that make a mix'd Number are less than an *Unit* of the superiour, and have always a known and cer-

certain Relation to them, (as 1 Shilling is a 20th Part of a Pound;) so they are in effect *Fractions*, and being summed up with a regard to that relation (as we shall learn) they are truly considered as *Fractions*, (or Numbers related to one another) in the Operation. But the relative Denomination (which in every Fraction is a Number) being suppress'd and understood, [tho' considered in the Operation] and each Species distinguished by proper Denominations, which of themselves express no such Relation; they are all considered as *Whole Numbers*: as indeed each Species is in the most strict and proper sense, considered as an Integer with respect to the lower. So that each of them, except the highest and lowest, is considered both as a *Whole Number* and a *Fraction*.

This Account of the Nature of *mix'd Numbers* might perhaps be sufficient; yet there are several Reflections, useful to such as would have complete Notions of Things, concerning the Nature of the various Kinds of *mix'd Numbers*, that may be very proper in this Place: for tho' it may be thought a Digression from the business of *Arithmetical Operations*, yet it can never be impertinent to make useful Reflections upon the Subject of these Operations.

The Nature and Design of Society has introduced among Men a Necessity of exchanging such things as are the Product of their different Applications and Labour; for every Country does not produce, nor every Man apply himself to every thing: Now, whatever way this Exchange is made; whether things are valued by their real Use, or by Fancy, there is always some Equality supposed, or made by agreement of Parties, betwixt certain Quantities of one thing and another. And that Commerce might be regular and certain, it was necessary to constitute some fixed and standard Quantities under certain and constant Names; otherwise Men could never be able to treat about these Exchanges unless they were together, and the Subjects were immediately before them; and even then not without great Inconveniency. Again, because Men need less and more on different occasions, it was necessary there should be various Quantities of every Kind, which differing only as *less* or *more*, it was convenient that each (or several) of the greater should contain a precise Number of the lesser as distinct and certain Parts of them, whereby subordinate Quantities coming under one general Name, constitute one kind of *mix'd Number*; the several Parts or Denominations of which we call the several Species of that Kind.

The more common Subjects of these *mix'd Numbers* are the external sensible Objects which we see and feel, and which in general we call Bodies.

Now the Quantity of Bodies can only be considered in two respects, either as to their *Bulk*, i. e. the external Measure of *Length*, *Breadth*, and *Thicknes*; or their *Weight*: And to compare different Bodies in these respects, there must be certain common standard *Measures* and *Weights* to which all others are compared. In some Bodies only one Dimension, which we call *Length*, is considered; because the other two are either inconsiderable in themselves; or rather, because in a comparison of more and less of these things, the other Dimensions are equal, (or supposed to be so;) and the Dimension chosen is that which admits of most Degrees: so that the more and less are here according to the *Length*. Hence proceed what we call the *Measures of Length*. See the following Tables of *mix'd Numbers*. Again, in some, Length and Breadth are both considered; and from this proceed the *Superficial* or *Square Measures*. Others are measured in all their three Dimensions; hence the *Solid* and *Cubical Measures*: under which may be comprehended what we call the *Measures of Capacity*, by which are measured the Quantity of *Liquids*, and of all such things as consisting of small Particles either altogether distinct, or cohering very loosely, cannot be measured singly, nor given out by Number; but are considered according to the *Bulk* they make, when being laid together they appear as one continuous Body; for Example, Corns, and Meal, or any solid Body reduced to Powder.

Again, other things are more conveniently measured by their *Weight*.

In the next place we must observe, That some things are exchanged by Number, the Individuals (which must all be of one Species of things) being really separated and distinct;

stinct; and which having neither superiour nor inferiour Species, are not valued by Weight or Measure; or one of them being so valued, the rest are supposed to be equal; or the things are such, as cannot be weighed or measured. In this manner are Cattle, and innumerable other things bought and sold by *Tale*, (in the common Phrase;) and for these there is no distinct Order of *mix'd Numbers*. But there is also a kind of *mix'd Numbers* constituted for some things that are exchanged by *Tale*; the Species of which are called *Gross* and *Dozen*, &c. Now here the Species are not any real continued Quantities, but certain Numbers distinguished by particular Denominations, which therefore require no Standards, but to have a true Idea of the Number they are affixed to. Whereas in other things, the several Denominations give us immediately the Idea of some continued Quantity; and we apply Number to them only by an arbitrary Subdivision into Parts: So that we may conclude, that *mix'd Numbers* arise more generally and properly from the imagined Parts of continued Quantities, either *Solids*, *Superficies*, or *Lines*.

As to *COINS*, observe, That they are properly measured by Weight: But the Weight being ascertained by publick legal Marks, their Species have not the proper Denomination of Weight; and therefore we don't ordinarily talk of them as things *weighed*; yet when there is any suspicion of false Weights, they are compared to some standard Weights.

As Numbers are not only applied to Bodies, and their imagined Parts, but also to every thing that is capable of *more* and *less*; as to the conceivable Parts of Time; so we have also from this last a particular kind of *mix'd Number*.

Again observe, That for different things we have different kinds of Weights and Measures; so we have *Troy Weight* and *Averdupoise Weight*, &c. We have different Measures for *Corn*, *Beer*, *Wine*, &c. Wherein these different *Weights* and *Measures* coincide and agree, or what the Relation betwixt them is, and by what means their Standards were first settled, is not so strictly the business of this Work to consider. The Statutes explain and determine these things; and perhaps Custom only is the Foundation of some of them.

The last thing I shall observe upon this Subject, is, That of the Denominations of *Coins*, *Weights*, and *Measures*, some are merely imaginary, *i. e.* are not Names of any one real distinct Quantity, but of some possible Quantity supposed equal to a certain Number, or a certain Part of some real standard Quantity. So for Example, a *Pound of Money* is an imaginary Quantity equal to 20 Shillings. A *Last* is an imaginary Quantity equal to 12 Barrels. And again, you must observe, that there are many more Denominations known, and used upon different Occasions in treating or speaking of these things, than are convenient or ordinarily used in keeping Accounts. In the following *Tables* I shall give you a full account of all the Kinds and Denominations that are commonly known; and distinguish those that are used in keeping Accounts.

TABLES of the most common COINS, WEIGHTS, and MEASURES,
[Real and Imaginary] of GREAT BRITAIN.

English Money.

4 Farthings	}	1 Penny.
4 Pence		1 Groat.
6 Pence		1 Tester.
12 Pence		1 Shilling.
5 Shillings	}>=	1 Crown.
6 Shillings + 8 Pence		1 Noble.
10 Shillings		1 Angel.
13 Shillings + 4 Pence		1 Mark.
20 Shillings	}	1 Pound.

The *Real Coins* now Current and commonly known, are these:

1. Of Copper-Money; a Farthing, and a Halfpenny.

2. Of Silver-Money; a Penny, Twopence, Fourpence, Sixpence, a Shilling, Half a Crown, a Crown.

3. Of Gold-Money; Half a Guinea = 10 Shillings + 6 Pence; a Guinea = 21 Shillings, or 1 Pound + 1 Shilling.

There are many other Gold-Coins, and some Silver, but not very common.

Accounts are kept in Pounds, Shillings, Pence, and Farthings; which are marked by these Characters, $l : s : d : q$ or f . Whose Relations I mark over them, thus;

$$\begin{array}{ccccccc} 20 & 12 & 4 & i. e. & 4f & = & 1d. \\ l : s : d : f. & & & & 12d & = & 1s. \\ & & & & 20s & = & 1l. \end{array}$$

Observe, In *Scotland* they use the same Denominations, except Farthings, and 1 Pound *English* = 12 Pound *Scotch*. But they begin now to use *English* Money in their Accounts.

ENGLISH WEIGHTS.

Troy Weight.

24 Grains	1 Penny-weight.
20 Penny-weight	= 1 Ounce.
12 Ounces	1 Pound.

Apothecary's Weight.

20 Grains	1 Scruple.
3 Scruples	= 1 Dram.
8 Drams	= 1 Ounce.
12 Ounces	1 Pound.

Accounts are kept in the same Denominations, marked thus:

$$\begin{array}{cccc} 12 & 20 & 24 & \\ \text{lb} : \text{oz} : \text{dw} : \text{gr.} \end{array}$$

Marked thus.

$$\begin{array}{cccc} 12 & 8 & 3 & 20 \\ \text{m} : \text{z} : \text{s} : \text{d} : \text{gr.} \end{array}$$

Averdupoise Weight.

4 Quarters of } a Dram	1 Dram.
16 Drams	1 Ounce.
16 Ounces	1 Pound.
14 Pound	= 1 Stone.
28 Pound	1 Quart. of a Hund.
4 Quarters of } a Hundred	1 Hundred Weight.
20 Hundred	1 Tun.

In keeping Accounts, this Weight is subdivided into two Kinds, called *Averdupoise Weight, the Greater, and the Lesser.*

The Greater comprehends these Denominations, Tun. Hundred Weight. Quarter.

Pound. Marked thus; T. C. Qr. lb. But the last 3, viz. C. Qr. lb. are sufficient.

The lesser comprehends these; Stone. Pound. Ounce. Dram. Quarter. Marked

$$\begin{array}{cccc} 14 & 16 & 16 & 4 \\ \text{thus : St. lb. oz. dr. qr.} \end{array}$$

Observe, In *Scotland* the Stone is commonly reckoned 16 Pounds.

The *Original* of all Weights in *England* was a *Corn* of Wheat taken out of the Middle of the Ear, and well dried; of which 32 made one Penny-weight; instead of which, they made afterwards another Division of the Penny-weight into 24 Grains. Mr. *Watd.* (in his *Young Mathematician's Guide*;) cites a Statute of *Edward III.* by which there ought to be no Weight used but *Troy*. But Custom, says he, afterwards prevailed in giving larger Weight to coarse and drossy Commodities, and thereby introduced the Weight called *Averdupoise*. And as to the Proportion betwixt *Troy* and *Averdupoise* Weight, he says, That by a very nice Experiment he found that 1 Pound *Averdupoise* is equal to 14 Ounces, 11 Penny-weight, 15 and $\frac{3}{4}$ Grains *Troy*. So that neither the Ounce nor Pound are the same.

By *Troy* Weight are weighed *Jewels, Gold, Silver, and Bread.*

By *Averdupoise* Weight are weighed all *Grocery Wares.*

The Apothecary's Pound is *Troy* Weight: but instead of subdividing the Ounce into *dw.* they divide it into Drams and Scruples.

Sheeps Wool Weight has these Denominations; 7 Pounds = 1 Clove: 2 Cloves = 1 Stone: 2 Stones = 1 Tod: $6\frac{1}{2}$ Tods = 1 Wey: 2 Weys = 1 Sack: 12 Sacks = 1 Last.

LIQUID

LIQUID MEASURE.

Wine Measure.

2 Pints	1 Quart.
4 Quarts	1 Gallon.
42 Gallons	1 Tierce.
1 $\frac{1}{2}$ Tierce	1 Hoghead.
1 $\frac{3}{4}$ Hoghead	= 1 Punchion.
1 $\frac{3}{4}$ Punchion or } 2 Hogheads }	1 Butt or Pipe.
2 Butts or Pipes	1 Tun.

Ale and Beer Measure.

2 Pints	1 Quart.	} A Firkin of Soap and Her- rings are the same with that of Ale.
4 Quarts	1 Gallon.	
8 Gallons Ale	} = 1 Firkin	
9 Gallons Beer		
2 Firkins	1 Kilderkin.	
2 Kilderkins	1 Barrel.	
1 $\frac{1}{2}$ Barrels	1 Hoghead.	

Which may for Accounts be reduced to these; Tun. Hoghead. Gallon. Quart. Pint. Thus Marked:

T : hd : gal : qt : pt.

Or it may be sufficient to use hd : gal : pt.

Observe also, That all Spirits, Mead, Perry, Cyder, Vinegar, Oil, and Honey are measured as Wine. Again, 18 Gallons make 1 Runlet; and 31 $\frac{1}{2}$ Gallons make a Wine or Vinegar Barrel.

Mr. *Ward* says, This Distinction of the Ale and Beer Measures are now used only in *London*; but in all other Places of *England* it is by a Statute of Excise made in the Year 1689, without distinction 8 $\frac{1}{2}$ Gallons to 1 Firkin. And this Measure may be reduced to these Denominations, viz.

hd : gall : qt : pt. for Ale.

hd : gall : qt : pt. for Beer.

Or, according to the last Account for both,

hd : gall : qt : pt.

There is also another way of keeping Accounts, especially in the Affairs of the *Revenue*, as the *Excise*; where they make the lowest Denomination a Cubical Inch, (*i. e.* a Measure 1 Inch long, 1 Inch broad, and 1 Inch deep:) And then of Wine Measure, 231 Cubick Inches make 1 Gallon. In Ale and Beer Measure, 282 Cubick Inches make 1 Gallon; and you may chuse as many of the superiour Denominations in your Accounts as you please.

As to the Original of Liquid Measure, it is from *Troy Weight*. Thus; 8 lb *Troy Weight* of Wheat gathered out of the Middle of the Ear, and well dried, is, by the old Statutes of *Henry III.* &c. ordained to be a Gallon of Wine Measure; neither were any other Measures allowed, tho' Time and Custom has introduced others. Mr. *Ward* mentions an Experiment he was witness to at *Guildhall*, before the Lord-Mayor of *London* and others; whereby it was found that the old Standard Wine Gallon contained exactly 224 Cubical Inches; tho', says he, for several Reasons, the supposed Content of 231 Inches was continued.

Observe, In *Scotland*, the common Denominations of Liquid Measure are these: Hoghead. Gallon. Pint. Mutchkin. Gill. and 4 Gill = 1 Mutchkin; 4 Mutchkins = 1 Pint; 8 Pints = 1 Gallon; and 16 Gallons = 1 Hoghead. They also call 2 Mutchkins, 1 Chopin; and 2 Pints 1 Quart. The *English* Pint is a very little larger than a *Scotch* Mutchkin. But the Excise in *Scotland*, since the Union of the two Nations, is calculated upon *English* Measure.

DRY MEASURE, *called*
also CORN MEASURE.

2 Pints	1 Quart.
2 Quarts	1 Pottle.
2 Pottles	1 Gallon.
2 Gallons	1 Peck.
4 Pecks	1 Bushel, Corn.
5 Pecks =	1 Bushel, Water.
4 Bushels	1 Coomb.
2 Coombs	1 Quarter.
4 Quarters	1 Chalder.
5 Quarters	1 Tun or Wey.
2 Weys	1 Last.

For Accounts, these Denominations are sufficient;

4 8 4 16
Ch : qr : bush : pk : pt.

I have taken this Table as it is in *Wingate* and others; but *Ward* says, 10 Quarters = 1 Wey, and 12 Weys = 1 Last of Corn.

As in Liquid Measure, so in dry, the lowest Denomination used in the Calculations of the Revenue is a Cubic Inch, whereof $268 \frac{4}{5}$ make 1 Gallon: For the *Winchester* Bushel with a plain round Bottom and equally wide from Top to Bottom, is $18 \frac{1}{2}$ Inches wide, and 8 Inches deep; whence follows by

Calculation, that $268 \frac{4}{5}$ Cubical Inches make 1 Gallon.

The common Denominations of Corn Measure in *Scotland*, are Chaldron. Boll. Bushel. ¹⁶ ⁴
⁴ ⁴ Peck. Quarter. But they are different Measures from the *English* of the same Name.

MEASURES of LENGTH.

	1ft.
12 Inches	1 Foot.
3 Feet	1 Yard.
45 Inches	1 Eln.
2 Yards =	1 Fathom.
$5 \frac{1}{2}$ Yards	1 Pole or Perch.
40 Poles	1 Furlong.
8 Furlongs	1 Mile.

For Accounts use these Denominations;

8 220 3 12
Mile : furl : yd : feet : Inch.

2d.
4 Nails = 1 Quarter
4 Quarts = 1 Yard.

Marked
4 4
yd : qr : in.

The Original of Long Measures is from a Corn of Barley, whereof 3 taken out of the Middle of the Ear, and well dried, make 1 Inch; and therefore 1 Barley-Corn is the least Measure, but not used in Accounts.

TIME.

60 Seconds	1 Minute.
60 Minutes	1 Hour.
24 Hours	= 1 Day.
365 Da. + 5 ho. + 48 min. + 57 sec.	1 Year.

then any Number of Days may be again reduced to Years, by dividing them by 365 Days, 5 Hours, 48 Minutes, 57 Seconds, as will be afterwards taught. In Astronomical Calculations there is a necessity to be thus exact: But for common Uses we may neglect the 5 Hours, 48 Minutes, 57 Seconds, and make 365 Days = 1 Year. Or also casting away 1 Day + 5 Hours, &c. we may call 364 Days = 1 Year: And make this Division of the Year, *viz.*

Because 1 Day is not an *aliquot* Part of a Year, therefore all these Denominations cannot conveniently be mixed in Accounts. And we may chuse to make Days the greatest Denomination in Accounts; and

60 Seconds	1 Minute.
60 Minutes	1 Hour.
24 Hours	1 Day.
7 Days	= 1 Week.
4 Weeks, or } 28 Days	1 Month.
12 Months	1 Year.

But to make 365 Days = 1 Year, and use no Denomination betwixt Year and Day, is the better way in Calculations of Interest, where it occurs most; as you'll find afterwards: yet Months and Weeks make a convenient Division of Time.

Of SUPERFICIAL or SQUARE MEASURE.

16 Square Quarters } of an Inch	1 Square Inch.
144 Square Inches	1 Square Foot.
9 Square Feet	= 1 Square Yard.
30 $\frac{1}{4}$ Square Yards	1 Square Pole.
40 Square Poles	1 Rod of Land.
4 Rods	1 Acre.

Observe, Square Measure is that which is as long as broad; and therefore as 4 Quarters make 1 Inch in Length, so a Surface 1 Inch long and 1 Inch broad is divisible into 16 Parts, each $\frac{1}{4}$ Inch long and broad; and so of the rest. The Reason of which will be understood after you know what Multiplication is.

For small Surfaces the Denominations of inch. foot. yd. are enough. And for Land these of Acre. Rod. Pole. Also when we say so many square Feet or Yards, &c. it were the same thing to say so many Feet or Yards long, and one broad. And thus a Rod is 40 Poles (or 220 Yards) long, and 1 Pole (or 5 $\frac{1}{2}$ Yards) broad; which is also 1210 Yards long, and 1 broad.

SCHOLIUM, relating to the following Applications.

Those who would be very nice and scrupulous as to the Method and Order of bringing in the following *Applications*, would first explain all the four fundamental Operations of *Addition*, *Subtraction*, *Multiplication*, and *Division*, as applied to simple Numbers, before they say a word of mixed Numbers; and they have this Reason for it, *viz.* Because the Addition of mixed Numbers is indeed not the simple Effect of Addition, but of that and Division too: and therefore, according to the strictest Method, ought to be brought as a mixed Application of both. It is true, that according to the Order we have hitherto followed, (the general Rule of Division being already taught) we may propose any Rule with a Division to be performed; for this cannot be called the proposing to one the doing of a thing which he has not yet learned: yet still it will have thus much of *Disorder* in it, that we shall anticipate something that does more immediately and properly belong to the Application of *Division*. But that being simple, and the Reason of it very obvious, I have chosen rather to explain the Practice of *Simple* and *Mixed Numbers* one immediately after the other, in each of the four Operations. And to serve those who incline to learn the Addition of Applicate Numbers before they learn Division of Abstract Numbers, I have also explained the Method necessary to be used in that Case.

§. 2. ADDITION of APPLICATE NUMBERS.

CASE I. To add Simple Numbers all of one Denomination.

RULE. This is done in all respects as *Abstract Numbers*, these being no other than *Numbers applicable to any kind of thing*. So that a particular Application can never alter the Rule and Reason of Operation. And that they ought to be all of one Denomination, is also plain. But see the following *Scholiums*.

SCHOLIUMS.

L. Sterling.	Years.
2468	347
7890	256
5678	789
2245	563
L. 18381	Sum Y. 1955

1. That is called a *Proper Addition*, when Numbers are so added together as that their Sum is a new Number distinct from either of them; and that *Applicate Numbers* may be added in this manner together, the *Units* of each that are to be added must be of the same Value and Denomination, or applied to the same Species of things, else the Sum

can have no particular Denomination, and so be of no use in Practice. So 4 Men and 3 Books make 7 things; but are not either 7 Men or 7 Books.

Again, Numbers of different Denominations are said to be added *improperly* when each Number is distinctly represented by itself with some Mark or Word for Addition; as for Example, 3*l.* and 4*s.* or 3*l.* + 4*s.* or more simply 3*l.*: 4*s.* and such Additions constitute *mixed Numbers*. But still it is to be minded, that *applicate Numbers* cannot be added, even improperly, *i. e.* to make one *mixed Number*, unless they have certain Relations to one another, so that a certain Number of one kind is equal to one of another. And for the same Reason when there are several *mixed Numbers* to be added, the Numbers of each Species must be separately and distinctly added together, as in the following Case: Concerning which, carefully observe the following Article.

2. The Sum of several *mixed Numbers* may be found and expressed after two very different manners. (1.) We may add the Numbers of every Species by itself into one complete Sum; and express these several Sums distinctly and separately. Or, (2.) Regarding the mutual Relations of the several Species, the total Sum may be found and expressed in a more simple manner; so that there shall be no Number in it of any inferior Species but what is less than an *Unit* of the next above. [The Value of the Sums of each inferior Species being expressed in Numbers of the next higher Species, gradually to the highest.]

Now the *First* is the only Method natural and proper to *Addition*; for the Answer it finds is the pure Effect of *Addition*; and is indeed only so many distinct Questions of simple Numbers added, without any dependence or regard to one another. As in the annex'd *Example*.

	<i>l.</i>	<i>s.</i>	<i>d.</i>
	24	14	10
	68	18	09
	352	06	11
	467	10	08
1st Method	911	48	38
2d Method	913	11	02

But the Answer found by the second Method, (which shall be taught immediately,) is the most Simple and Useful in Business; because it expresses the Whole in one Species (and that the highest) as near as possible; and so makes the Comparison of different Quantities more simple and easy: yet the Operation is more complex; for it is the mixed Effect of *Addition* and *Division*. I have in this *Example* expressed the Sum both ways. How the second is performed, you learn in the following *Rule*.

Lastly, Observe, That tho' the Numbers to be added are all of one Denomination, and so far belong to the first *Case*; yet if they are of such kind of things as have superior Species, the Answer may, in some Cases, be found and expressed two ways; either in their own Species by simple Addition, as in *Case* 1. or regard may be had to the superior Species: And then, if their Sum taken in their own Species is greater than an Unit of the Superiour, the total Value of the given Numbers may be expressed in the superiour Species, as far as it reaches. And so it will be either a simple Number of some one superiour Species, or a mix'd Number of several Species, (the Rule for the performing of which, is contain'd in that of the next *Case*.) For *Example*, several Numbers of *d.* or *sb.* being proposed to be added, we may find the Sum all in *d.* or *sb.* or in *l. s. d.* as far as the Value will reach. So if we take the Shillings of the preceding *Examp.* for the given Numbers of a Question, the Answer is either 48 *s.* or 2 *l.* 8 *s.* which you'll find equal to it by the Rule given in the following *Case*.

CASE II. To add MIX'D NUMBERS.

Rule 1. In every Line of mix'd Numbers let the Species be distinctly separated, and in order to this, write first down the Names (or Characters) of the Species, [the highest on the Left-hand, and the rest in order toward the Right] and then write every Line of Numbers in order under these, the Number of each Species under its own Name; and in every Species observe duly the Order of places of each Figure.

2. Beginning at the lowest Denomination, add all the Numbers of that Column together (as simple Numbers) and when you have found the Sum, you must find how many Units of the next superiour Denomination it's equal to: Thus; Divide it by that Number of the Species added, which is equal to an Unit of the next above; what remains in the Division write down under the Numbers added, as a Part of the total Sum which belongs to that Species, and the Number of the Quote take and add to the Numbers of the next Species. But if the Sum is less than an Unit of the next, set down what it is, and there is nothing to be carried to the next. Go thus thro' every Denomination, till you come to the last or highest, and write down the total Sum of that as it is, because it has relation to no higher; and all these Numbers, set down under every Denomination, make the total Sum.

Examp. of Money:

<i>l.</i>	<i>sb.</i>	<i>d.</i>	<i>f.</i>
476	14	11	2
6854	16	10	3
5923	08	06	2
640	10	09	0
832	00	11	3
7925	18	08	1
8894	19	10	3
35832	10	08	2

Thus, in the annex'd Example, the total Sum of the Farthings you'll find to be 14; then, because 4 Farthings = 1 Penny, I divide 14 by 4, the Quote is 3, and 2 remains, which is written in the Sum under Farthings, and the Quote 3 I carry to the Pence; and the Sum is 68 *d.* which I divide by 12 (because 12 *d.* = 1 *sb.*) the Quote is 5, and 8 remains, which is written down under *d.* and the 5 carried to the Shillings, whose Sum is 90, which I divide by 20 (because 20 *sb.* = 1 *l.*) the Quote is 4, and 10 remains, which is written under *sb.* and the 4 carried to the Pounds, whose Sum is 35832, making the total Sum 35832 *l.* 10 *s.* 08 *d.* 2 *f.*

The Reason of making the Divisions directed (which is the only new thing we have to account for) is plainly this, *viz.* Because 4 *f.* = 1 *d.* therefore as oft as 4 Farthings is contained in any other Number of Farthings, so many Pence is that Number of Farthings equal to: and the like Reasoning is good in all other Cases.

But for those who do not yet understand the Rule of Division (and even tho' they do) the following is a convenient Method.

To

To add MIX'D NUMBERS without the Rule of Division.

Rule. Begin at the lowest Species, and add the Numbers thereof together; not by single Columns, as you do simple Numbers, but take the whole Number that is in every Line together, and add them to one another, pointing when you come to such a Sum as is equal to, or greater than an Unit of the next higher Species, (but less than two such Units) and carry on the Excess, adding it to the next Number; and so thro' all that Species; setting down in the total Sum what Excess there happens to be after the last Point. Then for every Point carry 1 to the next Species, and go thro' all the Species in the same manner; but the highest you are to add by single Figures, as simple Numbers.

Examp.

<i>l.</i>	<i>s.</i>	<i>d.</i>
46	14	6
68	18	11
72	10	10
94	9	8
282	13	11

The Operation of this Example is thus: Beginning at the Pence of the lowest Line, I say $8 + 10 = 18$, which is $12 + 6$; therefore I make a Point at the 10, and carry forward the 6; thus $6 + 11 = 17$, which is $12 + 5$, this makes another Point at 11, and 5 to carry forward; then $5 + 6 = 11$, which being less than 12, I write it down: Then for the two Points I carry 2 to the Shillings; thus, $2 + 9 (= 11) + 10 = 21$, for which I make a Point (for the 20) at 10, and carry forwards the 1 over 20; thus $1 + 18 (= 19) + 14 = 33$, for which I make another Point at 14, and the Remainder 13 (over 20) is written down: then for these two Points I carry 2 to the Pounds, and add them as in *Case 1*.

Observe, We need not point the Shillings, but take this easy Method, *viz.* Add the first Column and write down what's over 10's, (as simple Numbers) then carrying the Number of 10's to the second Column, (or Place of 10's) sum it up, and if the Sum is an even Number, set down 0 and carry the half of the Sum to the Pounds; but if it's an odd Number, set down the odd 1, and carry the half of the Remainder.

The Reason of this Practice is plain, for two 10's make 20; and we may easily suppose any body, the least acquainted with Addition, can take the half of an even Number.

ANOTHER METHOD.

Some propose to make Tables, that may serve instead of Division; whereby, when the Sum of any Species is taken by itself (as simple Numbers) you may, by Inspection, find how many Units are to be carried to the next higher Species: The Method of which will be very obvious, by considering the following *Examp.* for Money, which is made by

TABLE for the Addition of Money.

<i>f.</i>	<i>d.</i>	<i>d.</i>	<i>sb.</i>	<i>sb.</i>	<i>l.</i>
4 =	1	12 =	1	20 =	1
8 =	2	24 =	2	40 =	2
12 =	3	36 =	3	60 =	3
16 =	4	48 =	4	80 =	4
20 =	5	60 =	5	100 =	5
24 =	6	72 =	6	120 =	6
28 =	7	84 =	7	140 =	7
32 =	8	96 =	8	160 =	8
36 =	9	108 =	9	180 =	9
40 =	10	120 =	10	200 =	10

simple Addition. Thus; Beginning at 4 *f.* write against it 1 *sb.* then $4 + 4 = 8$, and against it write 2; then $8 + 4 = 12$, and so on, still adding the last Sum to the first Number. The same way proceed in all other Species. And for the Length of the Table, you carry it on as far as you please; which, for long Pages or Columns of Numbers may require 40 Lines: But if you'll subdivide the Column into Parcels of about 12 or 20 Numbers, taking their Sums separately, and adding them together, then a Table carried so far will be sufficient.

The Use of this Table (and all others of the same kind) is obvious; for, having summed any Species, seek that Number in this Table (in its proper Column, if it does not exceed the greatest Number in the Table) and if that precise Number is not there, take the next lesser, and against it you have the Number of the next superiour Species contained in the Sum: which Number you are to carry to the next superiour, and take the Difference betwixt the Sum and that Number next less, which you are to write down under the Numbers added. I shall leave you to examine the preceding *Examples* by this Table, or make others for your Exercise.

Observe, Those who propose such Tables do it to prevent blotting of Accounts by pointing; for they design them for the more Ignorant, who can't do Division. But as I have said enough already to the Objection against pointing, I shall only observe, that any Accounts such Persons can be intrusted with, can't require so great Nicety as to make pointing a Fault. And I think 'tis plain, there is less Trouble with it in the Practice, and is even more convenient than to do the Work by Division (when one can do it so) because more simple.

EXAMPLES for the Exercise of ADDITION in MIX'D NUMBERS.

Money.				Troy Weight.				Averdupoise Weight, the Greater.		
l.	20 s.	12 d.	4 f.	lb.	12 3.	20 dw.	24 gr.	Ct.	4 qr.	28 lb.
346	14	08	3	4768	11	18	20	372	3	27
268	16	10	2	2345	10	15	23	468	2	20
4689	09	11	2	3689	08	10	18	593	0	10
7846	10	06	3	875	06	12	10	678	2	18
6320	00	04	0	762	04	19	06	976	3	19
25683	18	00	2	86	07	08	09	678	1	24
64	12	11	1	67	11	13	22	789	2	06
45219	17	05	1	12597	01	19	12	4558	1	12

Wine Measure.

Ton.	4 bd.	63 gal.	4 qt.	2 pt.
436	3	62	2	1
678	2	60	3	0
569	1	48	1	1
456	0	29	3	1
789	1	36	2	0
987	2	54	1	1
672	3	46	3	1
4591	1	24	1	1

Long Measure.

Mile.	8 furl.	220 yd.	3 f.	12 in.
3467	5	219	2	10
4567	7	184	1	11
5678	6	062	2	08
78967	4	009	0	09
56789	3	084	2	11
24608	2	147	1	10
35791	1	210	2	06
209871	0	80	0	05

In this *Example* of Long Measure, because 220 Yards = 1 Furlong, therefore in adding up the Yards, the easiest way is to add up the Column of Units, writing down what's over 10's, and carrying the Number of 10's to the other two Columns, sum them both together, pointing at every 22, or dividing the Sum by 22.

GENERAL SCHOLIUM, concerning the more special Application of the Rules of *Arithmetick*.

A good *Arithmetician* must be capable of something more than barely to perform any Operations with given Numbers, when the Question is simply proposed to Add, Subtract, &c. *i. e.* when he knows what Operation is to be applied, and to what Numbers. For, the great Art of Application lies in the Solution of such Questions as, to distinguish them from the other, I call *Mix'd* or *Circumstantiate* Questions, *i. e.* wherein no Operation is named, but we are left to find the proper Work from the Nature and Circumstances of the Question. Now, for this there are not any determinate and general Rules: it depends upon the good Sense and Judgment of the *Arithmetician*, whereby he can distinctly and perfectly comprehend the Nature and Circumstances of a Question. It supposes him to understand the Nature of the Subject about which the Question is; and lastly, to understand perfectly well the true general Import and Effect of the several simple Operations of *Arithmetick*. By which means he may know when the Reason of the Question requires such an Operation.

The more simple the Circumstances of a Question are, it will be the more easy; and where there is but one Operation to be applied, it will be always obvious: But, where a Variety of Circumstances occur, and several Operations become thereby necessary, the Difficulty increases; which Experience only can make easy. And therefore, as a proper Introduction to that Experience, I shall give you, after each of the Rules, some practical Questions, whose Solutions being considered, may help to guide the Judgment in like Applications.

The Effect of *Addition* being the Discovery of a Number, which is equal to certain given Numbers, taken all together; whenever the Sense and Reason of a Question shews that any given Numbers must be collected; or that the Number sought is equal in Value to several Numbers given, then Addition is the Rule; as in the following Examples.

MIX'D PRACTICAL QUESTIONS for ADDITION.

Quest. 1. A Father was 18 Years 4 Months old, (reckoning 13 Months to one Year, and 28 Days to one Month) when his eldest Child was born. Betwixt the eldest and second were 11 Months 10 Days. Betwixt the second and third were 3 Years 8 Months. When the third is 12 Years, 6 Months, 20 Days, how old is the Father? Answer 35 Years, 4 Months, 2 Days. For that all these Numbers ought to be added together, is manifest.

ye.	mo.	da.
18	04	00
	11	10
3	08	00
12	06	20
35	04	02

Quest. 2. I bought a Parcel of Goods, whereof the first Cost was 40 *l.* 10 *s.* paid for packing them 13 *s.* for Carriage 1 *l.* 6 *s.* 8 *d.* and spent about the Bargain making, 15 *s.* 6 *d.* What do these Goods stand me in all? Answer, 43 *l.* 5 *s.* 2 *d.*

l.	s.	d.
40	10	00
00	13	00
01	06	08
00	15	06
43	05	02

Quest.

Chap. 7. SUBTRACTION of *Applicate Numbers*.

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Quest. 3. There is owing me by the following Debtors, viz.
A. owes 20*l.* 15*s.* *B.* owes 100*l.* *C.* owes 56*l.* 10*s.* 8*d.* *D.* owes
 82*l.* 18*s.* 4*d.* What is the Amount of the whole? Answer,
 260*l.* 4*s.*

<i>l.</i>	<i>s.</i>	<i>d.</i>
20	15	00
100	00	00
56	10	08
82	18	04
260	04	00

Quest. 4. The Distance betwixt two Places is such, that if 3
 Miles and 5 Furlongs is taken from it, what remains is equal to
 8 Miles, 4 Furlongs and 100 Yards. What is the Distance, of
 these two Places? Answer, 12 Miles, 1 Furl. and 100 Yards.

<i>m.</i>	<i>furl.</i>	<i>yd.</i>
3	05	000
8	04	100
12	01	100

§. 3. SUBTRACTION of *APPLICATE NUMBERS*.

CASE 1. To subtract simple Numbers of one Denomination.

Rule. This is done as *Abstract Numbers*.

<i>l.</i>	Gallons.
Sub ^d . 468	Sub ^d . 72306
Sub ^r . 253	Sub ^r . 9462
Diff. 215 <i>l.</i>	Diff. 62844 Gallons.

CASE 2. To subtract *Mix'd Numbers*.

Rule 1. Having set the Subtractor orderly under the Subtrahend, with a due regard to the Places and Species; 2. Begin at the lowest Species of the Subtractor, and take the Number of that from its correspondent in the Subtrahend, and write down the Difference. Do the same with all the Species, and you have the Difference sought.

But if the Number of any inferior Species in the Subtractor, is greater than its Correspondent in the Subtrahend, then to this add a Number equal to an Unit of the next Species [for *Examp.* to *lb.* add 20, and to *d.* add 12] and subtract from that Sum; and then to the Number of the next Species in the Subtractor add 1, (or take 1 from the Subtrahend) and then subtract: And when you come to the highest Species, do as in simple Numbers, i. e. add 10 where you can't subtract.

Examp. 1. *Examp. 2.*

<i>l.</i>	<i>lb.</i>	<i>d.</i>	<i>l.</i>	<i>lb.</i>	<i>d.</i>
Sub ^d . 72	18	10	48	14	6
Sub ^r . 29	10	04	26	17	10
Diff. 43	08	06	21	16	08

The first *Example* is simple and easy. For the second, I do it thus, 10 (in the *d.*) from 6 cannot, but from 12 + 6 = 18, and 8 remains, to be set down in the *d.* Then 1 + 17 (in *lb.*) = 18, which from 14 cannot, but from 20 + 14 (= 34) and 16 remains, to be set down in *lb.* Then 1 + 6 (in *l.*) is 7, which taken from 8, 1 remains: Then 2 from 4, and 2 remains. So the total Difference is 21*l.* 16*s.* 8*d.*

The Reason of this Practice is sufficiently plain from the like Reasoning used for the abstract Rule.

SCHOLIUMS.

1. What was observed in Addition, is true here also, *viz.* That Number ought to be set down in any inferiour Species, but what's less than an Unit of the next; otherwise the Species are confounded: Yet still it's arbitrary to make any Species the highest, and then in it we write down any Number.

2. *Observe,* That this *Rule* supposes we can readily (or in our Mind) discover the Difference betwixt any Number of any Species less than an Unit of the next higher, and any other Number greater, but less also than such an Unit; or betwixt any such Number and a lesser increased by a Number equal to such an Unit. And to do this readily, requires a little Practice; but then the Operation for each Species may be performed separately, Figure by Figure, which will remove this Supposition; yet it will be more tedious Work; and we must by Practice acquire the Capacity which the Rule supposes. And to make it somewhat easier, take this Method: When the Subtrahend Figure is least, take the Subtractor Figure out of the Number to be added, and the Remainder add to the Subtrahend Figure; the Sum is the Number to be set down in the Difference. As in *Examp. 2.* above, I say 10 *d.* from 6 can't, but from 12, and 2 remains; which added to 6, makes 8 to be set down.

Other Examples in Subtraction.

Troy Weight.

lb.	oz.	dwt.	gr.
12	20	24	
342	: 08	: 10	: 06
84	: 03	: 15	: 20
258	: 04	: 14	: 10

Wine Measure.

bd.	gal.	qt.	pt.
33	4	2	
24	: 42	: 2	: 1
15	: 56	: 3	: 0
8	: 48	: 3	: 1

Long Measure.

yd.	qr.	na.
4	4	
18	: 2	: 0
9	: 3	: 1
8	: 2	: 3

MIX'D PRACTICAL QUESTIONS for SUBTRACTION.

Quest. 1. Having borrowed 20*l.* 13*s.* 4*d.* and paid thereof 8*l.* 16*s.* 8*d.* What's yet due? Answer, 11*l.* 16*s.* 8*d.*

l.	s.	d.
20	: 13	: 4
8	: 16	: 8
11	: 16	: 8

Quest. 2. Having bought 2 hund. Weight, and 3 qr. of Sugar; and sold thereof 1 hund. 2 qr. and 14 lb. what is yet unsold? Answer, 1 C. 14 lb.

C.	qr.	lb.
2	: 3	: 00
1	: 2	: 14
1	: 0	: 14

Quest. 3. A Father was 24 Years, 9 Months, 10 Days old when his eldest Son was born; and is now 56 Years, 3 Months, and 22 Days. How old is the Son? Answer, 31 ye. 7 mo. 12 da.

Ye.	mo.	da.
56	: 3	: 22
24	: 9	: 10
31	: 7	: 12

Quest. 4. What is that Sum of Money, which being added to 36*l.* 6*s.* 8*d.* will make 50*l.*? Answer, 13*l.* 13*s.* 4*d.*

l.	s.	d.
50	: 00	: 00
36	: 06	: 08
13	: 13	: 04

The

The following two Questions are mixed in *Addition* and *Subtraction*.

Quest. 5. Having borrowed 100*l.* and paid at one time 20*l.* 13*s.* 4*d.* at another time 33*l.* 6*s.* 2*d.* How much is yet due? *Answ.* 46*l.* 6*d.*

	<i>l.</i>	<i>s.</i>	<i>d.</i>
Borr ^d	100	00	00
	20	13	04
	33	06	02
Paid	53	19	06
	46	00	06

Quest. 6. There are 2 Casks of Sugar; the Weight of the one full Cask is 1 *C.* 2 *qr.* of the other 1 *C.* 18 *lb.* The Weight of one empty Cask is 25 *lb.* of the other, 1 *qr.* 7 *lb.* What is the Weight of Sugar in both Casks? *Answ.* 2 *C.* 14 *lb.*

	<i>C.</i>	<i>qr.</i>	<i>lb.</i>
	1	2	00
	1	0	18
Full Casks	2	2	18
	0	0	25
	0	1	07
Empty Casks	0	2	04
	2	0	14

§. 4. MULTIPLICATION of APPLICATE NUMBERS.

INTRODUCTION.

IN order to know what Variety there is in *Multiplication of Applicate Numbers*, we must first consider, What is a *Simple* and *Proper* Question of Multiplication; and I think it is plain, That this is a Question proposed with no other Circumstances, or in no other Form, but barely *To multiply one Number by another*; and then it is certain that the given Numbers must be such as are agreeable to the general Nature and Definition of that *Operation*; which being no other than the repeating or taking the one Number as oft as the other expresses, or contains *Unity*; it is manifest, that the Multiplier must always be an abstract Number, expressing simply the number of times that the Multiplicand is to be taken. Therefore the whole Variety in Multiplication of Applicate Numbers depends upon the *Multiplicand*; which may be either a *simple* or *mixed* Number, making only two different Cases.

Hence we see, That the proposing simply to multiply one Number by another, both Applicate, is absurd. For Example, to multiply 8*l.* by 3*l.* For what's that to say, *To multiply by 3 l.*? To multiply by 3 is intelligible; but what has the Name *l.* to do here? Now, because some have thought fit to propose such Questions without explaining the true Sense and Meaning of them, and given us very perplex'd Rules for solving them, which may be made more general and much easier, if the Sense of the Question is once rightly conceived; I shall here therefore make it appear more evidently, that Questions proposed in that simple manner, are Nonsense in Terms; by which means I shall lead you to the Sense that must be put upon all such Questions: and in its proper place you'll find the Rules for solving them.

Suppose then, that it is proposed to multiply 8*l.* by 3*l.* I should ask the Proposer, What he means by multiplying 8*l.*? He can make no other Answer, But that it is the repeating 8*l.* a certain number of times, (that being the simple and proper Definition of *Multiplication*.) Then I ask him, How many times he would have it repeated? And if he

he answers 3 *l.* I hope the Absurdity is manifest. He may indeed say, that he means to have it taken as oft as 3 *l.* contains 1 *l.* But then the direct simple Answer to my Question is 3 times, and not 3 *l.* And so the Denomination of *l.* applied to 3, does not belong to it as a Multiplier, and is only a certain circumstantial way of signifying how oft the *Multiplicand* is to be repeated: which is going round about to no purpose, when your Meaning can be expressed more simply; for you see it must still end in this, that the Multiplier is an abstract Number expressing only how oft the other Number is to be repeated.

Again observe, That the Authors of such Questions give us this for a Principle, That the two Terms must be both applicate to one kind of thing, as *Money, Weight, or Time*; so that they would very readily pronounce this Question, Nonsense, *viz.* To multiply 8 *l.* by 3 Days. But here their confused Notion of this matter will appear yet clearer; for this Proposition is every whit as reasonable as the other: Because if 3 *l.* cannot be a direct and reasonable Answer to that Question, *How oft?* till you explain it by saying, as oft as 3 *l.* contains 1 *l.* it is plain, that by the same Method, I may propose to multiply 8 *l.* by 3 Days; meaning to take 8 *l.* as oft as 3 Days contain 1 Day; which is equally good Sense as the other.

But further: Since an Explication is necessary; and something must be understood which is not directly expressed in proposing to multiply 8 *l.* by 3 *l.* it may as well signify the taking of 8 *l.* 60 times, [*i. e.* as oft as 3 *l.* contains 1 *sb.* which is 60 times;] for so I may explain it. *Again,* to multiply 8 *l.* by 3 *sb.* signifies, according to their meaning, taking $\frac{3}{20}$ Parts of 8 *l.* (because 1 *sb.* = $\frac{1}{20}$ of a *l.*) which is not purely a Question of *Multiplication*, but of that and *Division* too. But why may it not as well signify the taking 8 *l.* 3 times (*i. e.* as oft as 3 *sb.* contains 1 *sb.*) or 36 times, (*i. e.* as oft as 3 *sb.* contains 1 *d.*) All these meanings are equally reasonable; since the simple proposing to multiply 8 *l.* by 3 *sb.* limits it to none of them: for it has no meaning till it be explained; or, it has any one of these meanings indifferently. And this shews how ambiguous such Propositions are; or rather, no Propositions at all, till their meaning is thus cleared up and determined.

But now, after all this, you are to know that Questions may occur which are solved by *Multiplication*; yet the mix'd Circumstances of the Question be such, that all the given Numbers may be particularly applied; whereby the Reasonableness of such Propositions as I have here censured, may seem to be fairly accounted for. To which I answer, That in all such Cases, before we know what is to be done, we are obliged to reason upon the Nature and Circumstances of the Question, and by that means we discover, that some one given Number of things is to be taken as oft (*i. e.* multiplied) as some other contains an *Unit* of a certain Denomination. But then, as the *Multiplication* imports only the taking a number of things a certain number of times, so in the Operation, the *Multiplier* signifies only *how oft* the other is taken, (this being the proper and formal Notion of a *Multiplier*) tho' it is discovered by the Circumstance of a Number applied to some particular thing in the Question.

Again, the very same Number of things in different Questions will be considered in different Views, when it expresses the *Multiplier*, and so make different *Multiplications*; which shews that a simple Proposition of multiplying 8 *l.* by 3 *l.* or 3 *sb.* (and all Questions of this kind) have no determinate Meaning, or are plainly Nonsense. Nor is the Proposition less or more reasonable, tho' the two Terms are applied to different things which can be explained to the same meaning as the other, and occur as often in the Circumstances of mixed Questions.

I shall illustrate all this by particular Examples immediately; but shall first explain the Practice in simple Questions: which, according to the Distinction above-mentioned, consists of two Cases.

CASE I. To multiply a Simple Number.

RULE. This is to be done by the *General Rule for Abstract Numbers*; and the Product is an Applicate Number of the same Denomination with the *Multiplicand*.

Example 1.

$$\begin{array}{r} 7464 \text{ L.} \\ 8 \\ \hline 59712 \text{ L.} \end{array}$$

Example 2.

$$\begin{array}{r} 467 \text{ Years.} \\ 5 \\ \hline 2335 \text{ Y.} \end{array}$$

SCHOLIUMS:

I. Since any one of two Numbers may be made the Multiplier, and that it is sometimes more convenient to make it the one than the other, tho' that one happens to be the Applicate Number, (*i. e.* the Number to be multiplied) there is no matter; because it is but supposing the Denomination to be shifted and applied to the other Number, and then the Effect is the very same. For Example: To multiply 3*l.* by 48, I apply 3 as the Multiplier, as if it were 48*l.* by 3; whereby I make the Multiplier according to its proper Notion, Abstract; which gives the same Product and of the same Denomination: For since 48 times 3 = 3 times 48; if the Denomination of the Multiplicand is the same, the Product is in all respects the same.

II. *Multiplication* being in effect only a compendious Addition of Numbers equal among themselves, or the Repetition of the same Number, the Observations made in *Schol. 2.* after *Case 1.* in *Addition*, are applicable here also: *viz.* That the Product of a mixed Number may be found and expressed two ways; either, 1. By taking the Product of each Number by itself as so many distinct Questions and Applications of *Case 1.* which make a confused Answer, tho' the Work is simple Multiplication. Or, 2. Regarding the Relations of the Species, and expressing it in the highest as far as possible; which is considering it properly as a mixed Number, whereby we make a more simple and useful Answer, tho' the Operation is more complex; for it requires Division, the same way as Addition of mixed Numbers.

III. *Again*; When the Number multiplied is a simple Number of such things as have a superiour Species, the Product may be found also simply in its own Species; or may be expressed in superiour Species, as far as it reaches; according to the Method of the following Rule for mixed Numbers. But whereas in Addition we had an easy way to supply the Rule of Division by pointing, we cannot apply that Method here; and therefore Division is indispensible, if we would express the Product in its highest Species, unless we make use of the Tables described already for Addition; and then the Examples must be either very small Numbers, or the Tables exceeding large. But there is yet another way of expressing the Product of mixed Numbers simply without Division, which you'll find in the next Case. In the mean time, before we proceed to that, there is a special Class of mixed Practical Questions preparatory to it, which are Applications of the first Case, comprehended under this Title; *viz.*

REDUCTION from a Higher to a Lower Species; *i. e.*

Finding a Number of things of a Lower Species, (Denomination, or Value) Equivalent to a given Number of a Higher Species. For which, this is the Rule.

RULE.

RULE. Multiply the given Number by that Number of the Species to which you would reduce it, which makes an *Unit* of the Species reduced; the Product is the Number-sought.

<i>Examp. 1.</i>	<i>Examp. 2.</i>
48 <i>l.</i>	34 <i>lb.</i>
20	12
<hr/>	<hr/>
960 <i>sb.</i>	408 <i>oz.</i>
12	20
<hr/>	<hr/>
11520 <i>d.</i>	8168 <i>dwt.</i>
4	24
<hr/>	<hr/>
46080 <i>f.</i>	196032 <i>gr.</i>

Exam. 1. To reduce 48 *l.* to *sb.* it is = 960 *sb.* and this again to *d.* is = 11520 *d.* and again to *far.* is = 46080 *f.* According to the Operation in the Margin; each Step being in effect a new Question.

Exam. 2. To reduce 34 *lb.* *Troy Weight*; it is = 408 *oz.* = 8168 *dwt.* = 196032 *gr.*

The *Reason* of this Practice is obvious: For if 1 *l.* = 20 *sb.* then 48 *l.* is = 48 times 20 *sb.* or 20 times 48 *sb.* which is the same thing. (The like Reason will be found in other Cases.) Therefore you are to *observe*, That tho' the Denomination of *l.* is left with the given Number 48, as it stands in the Operation, yet it is performed in a quite different view; for we are to consider it not as the Multiplication of 48 *l.* by 20, the simple effect of which would certainly be, 960 *l.* But from the Reasons now explained, we consider it as the Multiplication of 48 *sb.* by 20, or of 20 *sb.* by 48; which is = 960 *sb.* And so of the rest. Which takes away the seeming Absurdity of naming the Product differently from the *Multiplicand*.

SCHOLIUMS.

I. If it is required to reduce a Number to a Species which is not immediately the next to it, as *l.* to *d.* We may either do it by Steps thro' all the intermediate Species, as above; or it may be done at one Multiplication, if we know how many *Units* of that Lower makes 1 of the Higher. For Example; To reduce *l.* to *d.* we multiply by 240; because 1 *l.* = 240 *d.* And for this purpose, it is ordinary to have Tables of Reduction, shewing how many *Units* of any Species (of common Use in Business) make 1 of any other; which are easily made by the preceding Rule, from the known Relations betwixt each Species and the next, which you have in the Tables of *Addition*. One or two Tables will be sufficient here to explain this Matter; and you may make the like for all other mixed Numbers, at your pleasure.

TABLES of REDUCTION.

1. Money.				2. Troy Weight.			
<i>l.</i>	<i>sb.</i>	<i>d.</i>	<i>f.</i>	<i>lb.</i>	<i>oz.</i>	<i>dwt.</i>	<i>gr.</i>
1	= 20	= 240	= 960	1	= 12	= 240	= 5760
	1	= 12	= 48		1	= 20	= 480
		1	= 4			1	= 24

The Use of these Tables is plain; for under every Denomination you have 1, and in the same Line the Number to which this 1 is equal of each inferior Species: So, for Example, to reduce *l.* to *f.* at once, the Multiplier is 960.

But you'll find that it will generally be as convenient to reduce any given Number to any inferior Species by reducing it gradually thro' all the Denominations, especially for this

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this Reason, that it were impossible to remember all these Tables, and too much trouble to turn to them upon every occasion; whereas the gradual Relations of the several Species are easily kept in mind. And lastly, the Reductions that most frequently happen, are of mixed Numbers, which necessarily must be done gradually; as in the following Article.

II. If a *mixed Number* is proposed to be reduced to the lowest Species expressed in it, [or to the lowest possible;] begin at the highest, and reduce it to the next Species, adding to the Product the given Number of that Species. Then reduce this Sum to the next Species, and so on thro' them all, taking in always the given Number of every inferior Species: As in the following Example; which needs, I think, no further Explanation.

Examp. 1.

$$\begin{array}{r}
 l : lb : d : f. \\
 724 : 17 : 09 : 2 \\
 20 \\
 \hline
 14480 lb. \\
 17 \text{ add} \\
 \hline
 14497 lb. \\
 12 \\
 \hline
 173964 d. \\
 9 \text{ add} \\
 \hline
 173973 d. \\
 4 \\
 \hline
 695892 f. \\
 2 \text{ add} \\
 \hline
 695894 f.
 \end{array}$$

Observe, The Numbers of the inferior Species may be taken into the Product of their Species without the pains of writing them down and adding them, by adding them Figure by Figure to the like places of the Product, as they are found in the Multiplying. You'll easily understand the Method by examining the following Examples. I shall only further observe, that it is best to take them all in upon the Multiplication with the Units place of the Multiplier, in case when this has two Places, you do the Work at length. But in the following Examples I have made the Product at once.

Ex. 2. *Troy Weight.*

$$\begin{array}{r}
 lb : oz : dw : gr. \\
 48 : 09 : 16 : 23 \\
 12 \\
 \hline
 585 oz. \\
 20 \\
 \hline
 11716 dw. \\
 24 \\
 \hline
 281207 gr.
 \end{array}$$

N.B. In this Ex. 2. as you may reduce the *dw.* at two Steps; multiplying first by 6, and then by 4, (because $6 \times 4 = 24$) you must mind that the 23 *gr.* are to be taken in with the last Multiplication.

Ex. 3. *Averdupoise Weight.*

$$\begin{array}{r}
 St : lb : oz : dr : gr. \\
 256 : 12 : 14 : 08 : 2 \\
 16 \\
 \hline
 4108 lb. \\
 16 \\
 \hline
 65842 oz. \\
 16 \\
 \hline
 1052474 gr.
 \end{array}$$

These Examples shew the Practice sufficiently; and we need give no other, but leave Examples of other kinds of Things to the Student's own Choice and Exercise.

You'll observe here also the great Difference betwixt multiplying a *mixed Number*, and reducing it; tho' this is performed by Multiplication: For multiplying it, is a Repetition of the Whole so many times, or finds a Number which contains every Part of the given mixed Number so many times; but reducing is only finding a Number equal to the given mixed Number in the lowest Denomination; in which every part of it is differently multiplied, and the last part not multiplied at all. So in the first Example, 724 *l.* is multiplied by 20, and 12, and 4 continually, that is by 960. But 17 *lb.* is only multiplied by 12 and 4, or 48. and 9 *d.* only by 4. The Answer of the Question being 695894 *f.* Whereas the Product of 724 *l.* : 17 *lb.* : 9 *d.* : 2 *f.* by any Number would be equal to that Number of times 695894 *f.*

N

CASE

CASE II. *To multiply a Mixed Number.*

RULE. Begin at the lowest Species of the *Multiplicand*, and having multiplied that Number, reduce the Product to the next Species; *i. e.* find by Division [in the manner already explained in *Addition of Mixed Numbers*] how many Units of the next superiour Species it is equal to, and what remains over; set what is over as a Part of the Answer of the Denomination multiplied. Then multiply the given Number of the next superiour Species, and to the Product add that Number to which the Product of the preceding Species was reduced; and reduce this Sum to the next superiour Species; marking the Remainder, or what is over, as a Part of the Answer of the Species multiplied; and go on thus thro' all the Species of the *Multiplicand*.

Examp. To multiply $236\text{ l} : 14\text{ sb} : 9\text{ d.}$ by 26.

In the annex'd Scheme you see the Method of the Work of this Rule; except that the Effect of the Reduction of the several Products is set down without the Operation by which it was done; these being supposed to be done a-part by themselves, and transferred to this Scheme. But you'll find another way immediately, wherein Division is used, and the whole Operation appears in one Scheme, without any confusion.

$$\begin{array}{r}
 \text{l} : \text{sb} : \text{d.} \\
 236 : 14 : 09 \\
 \hline
 6136 : 364 : 234 \text{ Prod. of each Species.} \\
 \quad 19 : 6 \dots = 234 \text{ d.} \\
 \quad 383 \dots \dots = 364 + 19 \text{ sb.} \\
 \quad 19 : 3 \dots \dots = 383 \text{ sb.} \\
 \quad 6155 \dots \dots = 6136 + 19 \text{ l.} \\
 \hline
 6155 : 03 : 06 \text{ Total Product.}
 \end{array}$$

Another Method without Division.

Reduce the *Multiplicand* to the lowest Species, as has been already taught; then multiply this Number by the given Multiplier, and the Product is the Number of that Species equivalent to the proposed Number of times the *Multiplicand*.

Operation.

$$\begin{array}{r}
 \text{l} : \text{sb} : \text{d.} \\
 48 : 16 : 8 \\
 20 \\
 \hline
 976 \text{ sb.} \\
 12 \\
 \hline
 11720 \text{ d.} \\
 42 \\
 \hline
 492240 \text{ d.}
 \end{array}$$

Examp. To multiply $48\text{ l} : 16\text{ sb} : 8\text{ d.}$ by 42, it is equal to 492240 d. as in the Margin.

This Answer is the only proper and natural Effect of Multiplication. And if it is required to know the Value of it in higher Species, this is properly a Question of Division, to be performed in the manner already explained; which is to divide by the Number of every Species which makes an Unit of the next above. But I shall refer you to Division to see the best and neatest Method of ordering the Operation. And here only observe these two things.

1. That with large Multipliers, this last Method (*viz.* of reducing to the lowest Species by Multiplication, and then to the higher by Division) will generally prove a more convenient Method than that of the first Rule. But, 2. If the Multiplier is a single Digit, or any Product of two Digits, the Work may, in most Cases, be easily performed according to the first Rule, without writing the Divisions: As in the following Examples.

Ex.

$$\begin{array}{r} \text{Ex. 1.} \quad l : sb : d. \\ 68 : 14 : 09 : 3 \\ \hline 483 : 03 : 8 : 1 \end{array}$$

$$\begin{array}{r} \text{Ex. 2.} \quad C : qr : lb \\ 37 : 3 : 18 \text{ by } 28. \\ \hline 148 : 2 : 16 \\ \hline 1040 : 2 : 00 \end{array}$$

Here I say, $3 \times 7 = 21$; which is 1 f. over 20, or 5 times 4: therefore I write down 1, and carry 5. Then $7 \times 9 = 63$, and 5 carried, is 68 d; which is 8 over 60, or 5 times 12: so I write 8 d. and carry 5. Then $7 \times 14 = 98$, and 5 is 103; for which I write 3, and carry 5, (for 5 times 20 is 100.) Then multiply the 68 l. and add the 5 from the Shillings.

For the 2d *Examp.* I resolve the Multiplier into 4 and 7. Then beginning with 4; I say, $4 \times 18 = 72$, which is 2 times 28 (=56,) and 16 over; or resolving the 4, I say $2 \times 18 = 36$, which is 28, and 8 over; consequently in 2 times 36 (=72) there are 2 times 28, and 2 times 8, or 16 over. The rest is easy. Then for the 7, I consider that 7 being the 4th Part of 28, and 4 the 4th Part of 16; therefore

7×16 , or 16 times 7, is equal to 4 times 28; for which I write 00, and carry 4. Then $7 \times 2 = 14$, and 4 carried is 18; for which I write 2 qr. and carry 4 C.

By such means as these, one may by Practice easily perform any Questions of this kind.

SCHOL. As to the Solution of other mixed Questions, there is no other general Direction can be given whereby to know when Multiplication is to be applied, but only this, *viz.* To consider, that the true Effect of Multiplication being the repeating of any Number, or taking it a certain number of times; therefore whenever the Sense and Reason of a Question requires that any given Number of things be repeated, or that a Number be found equal in Value to a certain given Number of things, repeated or taken as oft as some other given Number in the Question contains Unity; then Multiplication is the Work required. As in the following Examples.

Mixed Practical Questions in Multiplication.

Quest. 1. There is in each of 28 Bags, 44 l : 16 sb : 8 d. How much is in the Whole?
Ans. 1255 l : 6 sb : 8 d.

$$\begin{array}{r} l : sb : d. \\ 44 : 16 : 8 \\ \hline 179 : 06 : 8 \\ \hline 1255 : 06 : 8 \end{array}$$

Here the Nature of the Question plainly requires that 44 l : 16 sb : 8 d. be multiplied by 28, the Number of Bags; for if 1 Bag contains so much, 28 Bags must contain 28 times so much, which imports a Multiplication by 28; which is taken abstractly in the Operation, tho' it is applied to Bags in the mixed Proposition. As to the Manner of working, I have chosen 4 and 7 as *Factors*; because $4 \times 7 = 28$.

Quest. 2. At 3 l : 6 sb : 4 d. per Yard, what is the Value of 465 Yards? The Reason of this Question shews, that 3 l : 6 sb : 4 d. must be taken 465 times; or multiplied by 465. For the Value of 465 Yards must be 465 times as much as the Value of 1 Yard. And to do the Work, I reduce 3 l : 6 sb : 4 d. to d. it is = 796 d. which multiplied by 465, produces 370140 d. which is again equal to 1542 l : 5 sb. by Division. As you will learn afterwards.

$$\begin{array}{r}
 l : s : d. \\
 16 : 6 : 8 \\
 \hline
 98 : 0 : 0 \\
 18 \\
 \hline
 1764 \\
 7 \\
 \hline
 12348
 \end{array}$$

Quest. 3. There are 7 Chests of Drawers; in each of which are 18 Drawers; and in each of these are 6 Divisions; in each of which there is 16 *l* : 6 *s* : 8 *d*. How much Money is in the Whole? *Ans.* 12348 *l*.

It is plain there must be a Sum of Money equal to the continual Product of 16 *l* : 6 *s* : 8 *d*. by 6, 18, and 7.

Quest. 4. If 1 *l*. give 4 *s*. of Interest in any time; How much will 346 *l*. give in the same time? *Ans.* 1384 *s*. = 346 × 4 *s*. Here the 346 is applied to *l*. in the Proposition, but is an abstract Number in the Operation; which is not multiplying 4 *s*. by 346 *l*. but by the abstract Number 346.

If this Question be proposed, *viz.* If 1 *s*. yield 4 *s*. Interest; How much will 4 *l* : 10 *s*. yield in the same time? It is plain it must be 4 *s*. taken as oft as 4 *l* : 10 *s*. contains 1 *s*. *viz.* 90 times, (for 4 *l* : 10 *s*. = 90 *s*.) which makes 360 *s*. = 90 × 4 *s*. But this is not multiplying 4 *s*. by 4 *l* : 10 *s*. which would be an absurd Proposition.

Again, Suppose the Question were; If 1 *l*. yield 4 *s*. what will 4 *l* : 10 *s*. yield? The Answer is 6 *s*. equal to 4 *s*. taken as oft as 4 *l* : 10 *s*. contains 1 *l*. which is one and a half. But this, and all other Questions where Fractions come in, are not simple Questions of Multiplication. And as either of these Questions have an equal right to be called the Multiplication of 4 *s*. by 4 *l* : 10 *s*. it shews us how unreasonable such Propositions are, since it is the mixed Circumstances of the Question that determine how the Multiplication is to be made, which is different in different Circumstances.

§. 5. DIVISION of WHOLE and APPLICATE NUMBERS.

INTRODUCTION.

BEfore we enter upon this Application, we must consider the various Senses that may be put upon *Division*.

In the Definition, *Chap. 6.* there is but one Sense expressed; but there are other three ways of proposing a Question in *Division*, so dependent upon that in the Definition, that the same Number solves the Question in all the Senses in which it is a possible Question.

The first Sense is that in the Definition, *viz.* To find how oft one Number is contained in another. The second is to find, What Part of the Dividend the Divisor is equal to. The third is, To find a Number which is contained in the Dividend as oft as the Divisor expresses. The fourth is, To find a Number which is such a Part of the Dividend as the Divisor expresses or denominates.

Now it will easily appear, That the Answers to all these Questions, or the Impossibility of some of them in some Cases, is discovered by dividing according to the preceding Rule taken in the first Sense: Thus,

1. Let us first suppose, That the Division is without a Remainder, and all is plain: For the Number shewing how oft one Number is contained in another, (which is the first Sense,) does, from the nature of an *aliquot* Part, denominate what Part the Divisor is of the Dividend, (which is the second Sense.) Again, the same Quote is a Number contained in the Dividend as oft as the Divisor expresses; (which is the third Sense,) as has been shewn in the *Proof of Division*. And hence, lastly, it is such a Part of the Dividend as the Divisor denominates; (which is the fourth Sense.)

Examp.

Examp. $12 \div 3 = 4$, and nothing remaining; that is, 3 is contained 4 times in 12, (the first Sense.) And it is the 4th Part of 12, (the second sense.) Again, 4 is a Number contained 3 times in 12, (the third Sense.) And it is the 3^d Part of 12, (the fourth Sense.)

2. Suppose in the next place, That the Division has a Remainder; or, that the Divisor is not an *Aliquot* Part of the Dividend, [which includes that Case wherein the Divisor is greater than the Dividend.] Then the Question is Possible or Impossible, according to different Views and Limitations; as I shall here explain.

Let the Divisor be 3, and the Dividend 14; the Question is possible in the first Sense; and the Answer is 4 times, if we confine it to the number of times that the whole Divisor is contained in the Dividend: but taking it in a larger Sense, the Answer is $4\frac{2}{3}$. And in this View the Divisor may be greater than the Dividend: So if we ask how oft 14 is contained in 3, the Answer is $\frac{14}{3}$ Parts of a Time; the plain Sense of which is, that 3 contains $\frac{14}{3}$ Parts of 14.

In the second Sense, the Question supposes the Divisor is an *Aliquot* Part of the Dividend; and is therefore impossible when it is not so. But if we take a Part more largely for Part *Aliquot* or *Aliquant*, and ask what Fraction the Divisor is of the Dividend, then the Question is possible. But there is no new Question, strictly speaking; for it coincides with the first Question, changing the Dividend and Divisor, and taking *how oft* in the largest Sense. Thus; if we ask what Fraction 3 is of 14, the Answer is $\frac{3}{14}$; which is also the Answer, if we ask *how oft* 14 is contained in 3. And therefore I had rather in this Case call 3 the Dividend, and reduce it to the former Case; especially for this Reason, That the same Quote may be the Answer to all the really different Senses of the Question, while the Names of Divisor and Dividend are applied to the same Numbers.

In the third Sense, If we ask what is the greatest Number contained 3 times in 14; then if we limit it to a Whole Number, the Answer is 4. And if the Divisor is greater than the Dividend, the Question under this Limitation is impossible; as it also is if we should ask, what Whole Number is contained without a Remainder 3 times in 14; for this is contrary to supposition. But if we enlarge the Sense of the Question, and ask what Number, Whole, or Fraction, or Mix'd, is contained in the Dividend precisely as oft as the Divisor expresses, (so that the Product of the Divisor and Quote is equal to the Dividend,) the Question is always possible. Thus, as 3 is contained in 14, $4\frac{2}{3}$ times; so $4\frac{2}{3}$ is contained 3 times in 14; that is, 4 times any thing, and $\frac{2}{3}$ Parts of that thing is contained 3 times in 14 of that thing. For it has been already shewn, (in Chap. 6.

Schol. 4. after the Definition of *Division*;) that B times $\frac{A}{B}$ Parts of any thing, is equal to $\frac{A}{B}$ Parts of B times that thing; therefore as 3 is contained in 14, $4\frac{2}{3}$ times; so $4\frac{2}{3}$ is contained 3 times in 14. Or, because $\frac{14}{3} = 4\frac{2}{3}$; therefore as 3 is contained $4\frac{2}{3}$ times in 14, so $4\frac{2}{3}$ is contained 3 times in 14.

In the fourth Sense, the Question is impossible in Pure and Integral Numbers, when the Divisor is not an *Aliquot* Part of the Dividend; so because 3 is not an *Aliquot* Part of 14, there is no Number which is a third Part of 14; for if there were, 3 would be such a Part of 14 as that Number expresses. But taking the Question more largely, and admitting Fractions, it is in all Cases possible: So $4\frac{2}{3}$ Parts of any thing is a 3^d Part of 14, because it is contained in it 3 times.

As to the Circumstance which makes the third and fourth Sense possible, when the Divisor is not an *Aliquot* Part of the Dividend; it is remarkable, That the Subject of the Question is not pure Numbers, but such Quantities expressed by Numbers as are divisible, either really or imaginarily, into Parts or lesser Quantities: for in pure Numbers 14 has not a third Part; but considering the 14 as applied to some divisible Subject, the Quantity expressed by 14 has a third Part, which is expressed by $4\frac{2}{3}$; therefore the Question is possible only in *Applicate Numbers*.

From what's explain'd we see evidently, that as all the four Questions or Views of Division, are possible when the Divisor is an Aliquot Part of the Dividend; so when it is not an Aliquot Part, there are but three really different Questions; and which are all possible when the Subject of the Question is not pure Numbers, *i. e.* When we admit another Consideration than that of the Number of Things expressed, *viz.* their Divisibility into Parts or lesser Quantities: For then a Fraction comes naturally into the Answer, and makes a compleat Quote.

Now, from these different Views and Senses of Division, we learn what Variety there can possibly be in the particular Application of Numbers for a Question of Division; of which there can only be two *Cases*.

1. To make a Question in the first or second Sense, the Divisor and Dividend must both be applicate, and both to things of the same kind: And mutually, if the Divisor and Dividend are so applied, the Question admits only the first or second Sense; and the Quote is an abstract Number, shewing how oft the Divisor is contained in the Dividend, or denominates what Part the Divisor is of the Dividend, if there is no Remainder. For, as it is a reasonable Question to ask, How oft one Number of any kind of Things is contained in, or what Part it is of another of the same kind of Thing; so a Question being proposed in this manner, and either Divisor or Dividend being applied to a particular kind of Thing, the Nature of the Question imports, that the other is also applied, and to the same kind of Thing; since it's absurd to ask, How oft a Number of one kind of Thing, as 3 Pounds, is contained in a Number of another kind of Thing, as 14 Days?

2. To make a Question in the third or fourth Sense, the Dividend must be an Applicate Number, and the Divisor Abstract, denominating what Part of the Dividend the Quote is, or how oft it is contained in it: so that the Quote is a Number applicate to the same kind of Thing as the Dividend; the Part of any Thing being of the same Nature as the whole. Again, mutually the Dividend being considered as applied, and the Divisor as abstract, the Question can admit only of the third or fourth Sense.

That the Application must be ordered in the manner now explained, may be also deduced from the Connection and Dependence of Multiplication and Division: For, since in Multiplication the Product and one Factor must be applicate to the same kind of Thing, the other Factor being abstract; and in Division the Divisor and Quote produce the Dividend: it follows, that the Dividend, with the Divisor or Quote, are alike applied, the other being abstract.

Again, *Observe*, That in mix'd Questions it will happen that two Numbers which in the Proposition are applied to different Things, must be divided one by the other: But in this Case, you'll always find that the Number made Divisor is considered in the Operation, as abstract, denominating what part of the other the Nature and Reason of the Question requires to be taken. So that in all Cases it's true, that the Divisor is either abstract, or applied to the same kind of thing as the Dividend.

We shall next explain the simple Practice in Division of *Applicate Numbers*.

CASE 1. The Divisor and Dividend being both applied to the same kind of Thing.

Rule. Reduce, (if need be) the Divisor and Dividend to simple Numbers of one Name, (the lowest expressed in either Term;) then divide these Numbers by the general Rule. The Quote shews how oft the Divisor is contained in the Dividend, or what Part the Divisor is of the Dividend, when there is no Remainder.

Examp.

Examp. 1. 3*l.*) 24*l.* (8

Examp. 2. 3*l.*) 20*l.* (6 $\frac{2}{3}$

Examp. 3. 3*sh.*) 12*l.*
or,
3*sh.*) 240*sh.* (80

Examp. 4. years, mo. ye. mo. da.
3 : 4) 24 : 7 : 18
By Reduction at 13 Months to
a Year, and 28 days to 1 Month.

Examp. 5. 13 oz. 8 dr.) 128 lb.
By Reduction,
216 dr.) 30208 dr. (139 $\frac{184}{216}$
216
860
648
2128
1944
184

1204 days) 8950 da. (7 $\frac{528}{1204}$
8428
522

The *Reason* of this *Rule* is plain; for the Divisor and Dividend expressing Things of the same Value and Name, it is evident the Operation is to be managed as with abstract Numbers, by the General Rule; *i. e.* The Quantity expressed by the Divisor is contained in the Quantity expressed by the Dividend, as oft as the Divisor is contained in the Dividend, taken purely as Numbers. Again, If the Divisor and Dividend express Things of the same general Nature, which can be said to contain one another, then tho' they are not of one particular Species or Name, yet the Question is possible, only it requires that they be reduced to Numbers that express Quantities of one Species or Name; and then it is manifest, that the Division of these Numbers by the General Rule, gives the true Quote. So in *Exam. 1.* 3*l.* is contained in 24*l.* as oft as 3 in 24. But in *Exam. 2.* 3*s.* is oftner contained in 12*l.* than 3 in 12; for it is as oft as 3 in 240, the Number of Shillings equal to 12*l.* And, because the Divisor and Dividend are then only in a State to be managed as pure Numbers, when they are both simple Numbers of one Name, this shews the Reason of reducing mix'd Numbers.

CASE 2. The Dividend being *Applicate*, and the Divisor *Abstract*.

Rule. If the Dividend is a simple Number, greater than the Divisor, divide it by the General Rule; the Quote is a Number of the same things as the Dividend: and if there is no Remainder, the Operation is finished; but if there is a Remainder, reduce it to the next Denomination, and divide; and so on, as long as there is a Remainder, and any lower Denomination, and make a Fraction of the last Remainder. Thus you have the Answer in one Species or several; which is an *Applicate Number* contained in the Dividend as oft as the Divisor expresses, or is such a part of the Dividend as the Divisor denominates. (See *Ex. 1.*)

Again, If the Dividend is a simple Number, less than the Divisor, you must first reduce it to a lower Species, till it be equal to, or greater than the Divisor, and then divide and proceed with the Remainder as before. (See *Ex. 3, 4.*) If it's not equal to the Divisor in any Species, then the Answer is a Fraction of the given Species, whose Numerator is the given Dividend. (*Ex. 6.*)

Lastly, If the Dividend is a mix'd Number, you may do the Work two ways: Either (1.) Reduce it to a simple Number of the lowest Species, and then divide; so you'll have the Answer in that Species, (which may be reduced again to superiour Species by Division, as has been formerly explained, and will be more particularly by and by.) But it will be better to proceed in this manner: (2.) Begin with the Number of the highest Species in the Dividend; divide it, and reduce the Remainder to the next Species, taking in the given Number of that next Species; then divide; and so go on. (See *Ex. 2.*) But if the Number of the highest Species is less than the Divisor, reduce it, taking in the given Number of the next Species, and so on, till you have a Number equal to, or greater

greater than the Divisor: (Ex. 5.) And if that be not in any of the known Species, then the Answer is only a Fraction, whose Numerator is the Dividend, reduced to the lowest Species, and refers to an Unit of that Species. (See *Examp. 7.*)

Examp. 1.

$$\begin{array}{r} 3 \overline{) 14 \text{ l. } (4 \text{ l.}} \\ \underline{12} \\ 2 \text{ Rem.} \\ 20 \\ \underline{40 \text{ s. } (13 \text{ s.}} \\ 39 \\ \underline{1 \text{ Rem.}} \\ 12 \\ \underline{12 \text{ d. } (4 \text{ d.}} \end{array}$$

Examp. 2.

$$\begin{array}{r} 6 \overline{) 23 : 10 : 8 (3 \text{ l.}} \\ \underline{18} \\ 5 \text{ Rem.} \\ 20 \\ \underline{110 \text{ s. } (16 \text{ s.}} \\ 108 \\ \underline{2 \text{ Rem.}} \\ 12 \\ \underline{32 \text{ d. } (5 \text{ d.}} \\ 30 \\ \underline{2 \text{ Rem.}} \\ 4 \\ \underline{8 \text{ f. } (1 \frac{1}{2} \text{ f.}} \end{array}$$

Examp. 3.

$$\begin{array}{r} 24 \overline{) 18 \text{ l.}} \\ \underline{20} \\ 360 \text{ s. } (15 \text{ s.} \\ \underline{24} \\ 120 \\ \underline{120} \end{array}$$

Examp. 4.

$$\begin{array}{r} 42 \overline{) 16 \text{ l.}} \\ \underline{20} \\ 320 \text{ s. } (7 \text{ s.} \\ \underline{294} \\ 26 \text{ Rem.} \\ 12 \\ \underline{312 \text{ d. } (7 \text{ d.}} \\ 294 \\ \underline{18 \text{ Rem.}} \\ 4 \\ \underline{72 \text{ f. } (1 \frac{1}{2} \text{ f.}} \\ 42 \\ \underline{30} \end{array}$$

Examp. 5.

$$\begin{array}{r} 14 \overline{) 8 : 15} \\ \underline{20} \\ 175 \text{ s. } (12 \text{ s.} \\ \underline{168} \\ 7 \text{ Rem.} \\ 12 \\ \underline{84 \text{ d. } (6 \text{ d.}} \\ 84 \end{array}$$

Examp. 6.

$$3460 \overline{) 3 \text{ l. } (1 \frac{1}{48} \text{ of } 1 \text{ l. because } 3 \text{ reduced to Farthings is but } 2880 \text{ f.}}$$

Examp. 7.

$$\begin{array}{r} 3460 \overline{) 3 \text{ l. } 12 \text{ s.}} \\ \underline{20} \\ 72 (1 \frac{1}{48} \text{ of } 1 \text{ s.} \end{array}$$

The *Reason* of this Practice is plain: For, if we find any proposed Part of all the Members of which any Number or Quantity is composed, we have the like Part of the Whole; since the Whole is nothing else but all the Parts. And if the Dividend is less than the Divisor, yet if reduced to another Species, it is equal to, or greater than the Divisor; it's plain that the equivalent Number in another Species being divided by the same Divisor, gives the true Answer: Thus $18 \text{ l.} = 360 \text{ s.}$ therefore the 24th part of 18 l. is the 24th Part of 360 s.

SCHOLIUM. What's already done shews the Practice of Division, or the Solution of simple Questions, where the Proposition is directly and simply to divide one Number by another, (in any of the Senses above explained.) And as to the Solution of mix'd Questions, all the further Direction can be given for knowing when Division is to be applied, is to *consider* well the Effect of Division as above explained: Which may be reduced to two principal Views, *viz.* Finding how oft one Number is contained in another, or finding a proposed aliquot Part of a Number. Then, when the Reason and Nature of a

Question

Question requires, that you find how oft one Number, simple or mix'd, of any kind of Thing, is contained in another Number of the same kind of thing; or, that you find such a Part of any Number of things, as another Number denominates, or as an Unit of any Species of Things is of a certain given Number of the same Things; then is Division your Work, as in the following Examples.

MIX'D PRACTICAL QUESTIONS for DIVISION.

A special Class of such Questions is comprehended under Title of,

REDUCTION from a lower to a higher Species, i. e. To find a Number of things of a higher Species, equal in Value to a given Number of a lower Species; or, at least, to find the greatest Number of the higher contained in the Number of the lower, with what remains over of that lower Species: Supposing always, that an Unit of the lower Species is an aliquot Part of an Unit of the higher. For which this is the

Rule. Divide the given Number by that Number which expresses how many Units of the Species to be reduced, are contained in an Unit of the Species to which it is to be reduced. The Quote is the Number sought of that higher Species, and the Remainder is a Number of the Species reduced.

Thus you may reduce gradually from the lowest to the highest Species; or all at once to the highest, if the Number of Units of the lowest, which make one of the highest, is known. (As you may know by the Reduction Tables, explained in §. 4.)

In all the following Questions, I have performed the Divisions by the contracted Methods, explained in Chap. vi. §. 2.

Examp. 1. In Money. Here 74608395 *f.* being divided by 4, Quote 18652098 *d.* and 3 *f.* over; these *d.* divided by 12, Quote 1554341 *s.* and 6 *d.* over; these *sh.* divided by 20, Quote 77717 *l.* and 1 *s.* over. Wherefore 74608395 *f.* = 18652098 *d.* 3 *f.* = 1554341 *s.* 6 *d.* 77717 *l.* 1 *s.* 3 *f.* = 77717 *l.* 1 *s.* 6 *d.* 3 *f.*

By this you know how to explain the following Examples.

Examp. 2. Troy Weight.

$$\begin{array}{r} 24 \overline{) 4 \ 3476589426 \text{ grains.}} \\ \underline{4 \ 869147356} \\ 20 \overline{) 144857892 \text{ dw. : 18 gr.}} \\ \underline{12 \ 7242894} \\ 12 \overline{) 603574 \text{ oz. : 12 dw.}} \\ \underline{603574} \end{array}$$

Examp. 3. Long Measure.

$$\begin{array}{r} 12 \overline{) 46320689372589 \text{ inches.}} \\ \underline{3 \ 3860057447715} \\ 11 \overline{) 1286685815905 \text{ feet : 9 inch.}} \\ \underline{116971437809} \\ 20 \overline{) 5848571890 \text{ yd. : 00}} \\ \underline{8 \ 731071486} \end{array}$$

The Reason of Dividing in these Cases is obvious: For Ex. Since 4 Farthings = 1 *d.* therefore as many times as 4 *f.* are contained in any Number of *f.* so many *d.* is that Number of *f.* equal to. In which observe, that the immediate Effect of the Division is an abstract Number, shewing how oft 4 *f.* is contained in a greater Number of *f.* and we apply the Name of *d.* to the Quote, from the Reason of the Question, as now explain'd.

Quest. 1. 8 Men have equal Shares of a Stock of 146 *l.* 16 *s.* what is each Man's Share?
Answer, 18 *l.* 7 *s.* viz. the 8th Part of 146 *l.* 16 *s.* For the Nature of the Question plainly directs us to take an 8th Part. Where observe, that tho' 8 expresses a Number of Men in the mixt Proposition, yet in the Operation it is considered abstractly as the Denomination of that Part of 146 *l.* 16 *s.* which the Nature of the Question requires to be taken.

Quest. 2. A certain Number of Persons, each of whose Ages is 15 Years 3 Months (reckoning 13 Months to 1 Year) make up in all 182 Years, 10 Months; how many Persons are there? Answer, 12. *viz.* The Number of times that 15 Years, 3 Months, are contained in 182 Years, 10 Months.

$$\begin{array}{r} 198 \overline{) 2376} \quad (12 \\ 198 \\ \hline 396 \\ 396 \\ \hline \end{array}$$

Quest. 3. What is the Value of 1 Yard of Cloth. . . . 48 Yards cost 15 *l.* 10 *s.* 4 *d.* Answer, 6 *s.* . . . $2\frac{2}{3}$ *f.* = $\frac{1}{48}$ Part of 15 *l.* 10 *s.* 4 *d.*

$$\begin{array}{r} 48 \overline{) 15 \text{ } 10 \text{ } 4 \text{ } 0} \\ 8 \overline{) 2 \text{ } 11 \text{ } 8 \text{ } 2} \\ \hline 6 \text{ } 5 \text{ } 2 \frac{2}{3} \text{ or } \frac{1}{2} \end{array}$$

For it's plain, the Value of 1 Yard of 48 Yards, must be the 48th Part of the Value of the whole 48 Yards, which directs us to Division, or taking a 48th Part of 15 *l.* 10 *s.* 4 *d.* And so this 48, which in the mix'd Proposition is applied to Yards, is con-

sidered abstractly in the Operation; which is therefore not a Division of Money by Yards, which cannot be made in any Sense, but taking such a Part of the Money as 1 Yard is of 48 Yards, *viz.* a 48th Part.

Observe, Had it been proposed, in the last Question, to find the Value of 1 Quarter of a Yard, we may do it either by finding first the Value of 1 Yard, and then the 4th Part of this is the Value of 1 Quarter; or, by reducing 48 Yards to Quarters, which make 192; and taking the 192^d Part of the given Money. The Reason for which is the same as for the other Case.

The following Question requires all the four Operations of Arithmetick.

Quest. A Father left among 5 Sons an Estate, consisting of 500 *l.* in Cash; with 5 Bills, each of 48 *l.* 10 *s.* 6 *d.* He ordered 20 *l.* to be bestowed upon his Burial, and his Debts to be paid, amounting to 164 *l.* Then his free Estate to be divided in this manner, *viz.* The eldest Son to have the 3^d Part, and the other 4 Sons to have equal Shares. What is the Share of each Son? Answer 186 *l.* 4 *s.* 2 *d.* to the eldest; and 93 *l.* 2 *s.* 1 *d.* to each of the rest.

Operation.	l.
48 <i>l.</i> 10 <i>s.</i> 6 <i>d.</i>	20
5 Bills.	164
<hr/>	<hr/>
242 : 12 : 6	184
500 : 00 : 0 Cash.	
<hr/>	<hr/>
742 : 12 : 6 Total.	
134 : 00 : 0 deduced.	
<hr/>	<hr/>
3, 558 : 12 : 6 Free Estate.	
186 : 4 : 2 Eldest Son.	
<hr/>	<hr/>
4, 372 : 8 : 4 Remains.	
93 : 2 : 1 the Share of	
each of the other 4.	

SCHOLIUM. As Questions may be variously mix'd, so the Solution will depend upon a due Consideration of the several Parts of the Question, and what Operation each may require, according to the Sense and Effect of the different Operations in Arithmetick. But there are mix'd Applications of Multiplication and Division, which require other Rules and Directions, to know when and how they are to be made; these you'll learn in *Book 4.* and especially in *Book 6.* under the Name of *Proportion.* What has been already done in this Book, being sufficient for explaining the Nature of the fundamental Operations, and their more simple Applications in whole Numbers. For the Doctrine of Fractions, you have it in the next Book.

C H A P. VIII.

Containing the more particular Rules of the LITERAL ARITHMETICK, necessary in the following Parts of this Work.

I. *For ADDITION or SUBTRACTION.*

CASE I. If the Numbers to be added or subtracted are expressed all by the same Letter, multiplied by certain Numbers, as, $3a$, or $5b$, add or subtract the Coefficients (*i. e.* the Numbers by which the Letters are multiplied) and multiply the same Letter into the Sum or Difference, it is the Sum or Difference sought.

Examp. $3a + 5a = 8a.$ *Ex.* $2b + 3b + b = 6b.$ *Ex.* $5n - 2n = 3n.$
Ex. $4ab + 3ab = 7ab.$ *Ex.* $3ab - ab = 2ab.$

SCHOLIUM. In order to understand the other Cases, we must premise this *Observation*, viz. After the Addition or Subtraction of one Number to, or from another, we may suppose another added to, or subtracted from the preceding Sum or Difference; and another added to, or subtracted from the last Sum or Difference, and so on: Then is this whole Work expressed by setting the Numbers, or Letters representing them, in the same order, with the proper Signs of the several Operations betwixt them. Thus, if b is added to a , and from the Sum c is subtracted, and from this Difference d subtracted, and to this last Difference e added; the Result of all this is expressed thus, $a + b - c - d + e$. But again, *Observe*, That if the same Numbers can be added or subtracted in any other order, the final Result or Effect will still be the same; which, in all the possible Orders wherein the Operations can be made, is plainly the Difference betwixt the Sum of all these Terms that are added in the several Steps, and the Sum of all these that are subtracted: Because in whatever order any Numbers are added and subtracted, it's evident there is so much in whole added, as the Sum of all these that are added in the several Steps, and as much subtracted in whole, as the Sum of all that are subtracted in the several Steps: Wherefore, the final Result is the Difference of these Sums. Whence, again, this follows, That 'tis no matter in what order we place the several Terms of a mix'd Expression, if we always prefix the same Signs to the same Letters, and also take the Meaning of the Expression to be universally the subtracting all these Terms that have $-$ prefix'd, out of the Sum of all these that have $+$ prefix'd: So that when the Operations can be performed in a proposed Order, we may explain the Expression either according to that Order, or in the preceding general way, (if that is not the proposed Order.) And if they cannot be performed in the proposed Order, then we explain it after the general way, as the universal Meaning of all such Expressions; for tho' some may be explain'd another way, yet the final Result is always equal to this.

For *Example*, If b is greater than a , then $a - b + c$ can't be explained in the Order of these Letters and Signs; and if it is at all possible, it is so in this Order, $a + c - b$; yet it may represent the same thing standing thus, $a - b + c$, while we do not so much regard

the Order, as the general Meaning of the Signs, which is as if it were in this Order, $a + c - b$.

CASE II. If any complex Expression [whether it is a Sum, having all its Terms joined by the Sign $+$; or a Difference, having its Terms partly $+$, partly $-$] is to be subtracted from another, (expressed simply or complexly) the Difference sought may be expressed two ways.

Rule 1. By drawing a Line over the whole Terms of the Subtractor, and joining it to the Subtrahend with a Mark of Subtraction between them. Thus, the Difference of a and $b + c$ is expressed $a - \overline{b + c}$, signifying that b and c both, or their Sum, is taken from a , which is a quite different thing from $a - b + c$ without the Line, which would signify the Difference betwixt $a + c$ and b . Again, if the Subtractor is the Expression of a Difference, as $b - c$ from a , the Difference sought is expressed $a - \overline{b - c}$, signifying, that the Difference of b and c is taken from a ; i. e. That c is taken from b , and the Remainder taken from a , which is a different thing from $a - b - c$, which expresses the Difference of a and b, c both, i. e. That b and c are both taken from a .

Now, tho' this Method is sometimes convenient, yet it would often prove too general and indefinite, which is supplied by another Rule, whereby, from the simple Terms of the given Subtractor and Subtrahend, another Expression is found equal to the Difference sought. Thus,

Rule 2. Change the Signs of all the Terms of the Subtractor, and join them to the Subtrahend without a Line over them; and this expresses a Number equal to the Difference sought. Thus, if the Subtractor is a Sum, as $b + c$, the Difference of this and a is $a - b - c$ ($= a - \overline{b + c}$.)

The Reason is plain; for the Sum is subtracted, when all the Parts are subtracted one after another; since the Parts are equal to the Whole.

Again, If the Subtractor is a Difference, as if $b - c$ is to be taken from a , the Remainder is $a - b + c$ ($= a - \overline{b - c}$) which more directly represents the Difference betwixt $a + c$ and b , yet is equal to the Difference of a and $b - c$.

The Reason is plain; for by adding the lesser Term c , and then taking away the greater a , we take away as much as was before added, and also all that a exceeds b : Or, if b does not exceed a , we may first take b from a , and to the Difference add c ; for thus we restore all that was taken away, except so much as b exceeds c : And so both ways, the Thing really taken away is precisely what b exceeds c , or their Difference. If the Subtractor is a more complex Expression, or consists of more than two Terms added and subtracted, the Reason of the Rule is still the same, from what has been explained, viz. That such Expressions signify no more in Effect, than the Difference of the Sum of all that are added, and the Sum of all that are subtracted: Wherefore, by what's now shewn, all that are added in the Subtractor must, in expressing the Difference sought, be subtracted; and all that are subtracted in it, must be added: Thus, $a - \overline{b + c - d - e} = a - b - c + d + e$, or $a + d + e - b - c$.

SCHOLIUM. The last Case of this Rule may be considered as a Theorem, and expressed thus: If the Difference of two Numbers is subtracted from a third Number, the Remainder is the same, as if we added the lesser of these two Numbers to the third, and from the Sum subtracted the greater. So $a - \overline{b - c} = a + c - b$.

CASE III. If any complex Expression of a Difference is to be added to any other Expression, join them without changing their Signs, or any Line over them. *Examp.* If to a we add $b + c$, the Sum is $a + b + c$. Again, to a add $b - c$, the Sum is $a + b - c$. To $a - b$ add $c - d + e$, the Sum is $a - b + c - d + e$ ($= a + c + e - b - d$.)

The

The *Reason* of this *Rule* is evident, when the Expression added has all its Terms $+$, as in *Examp* 1. And if they are some $+$ some $-$, as in *Ex.* 2: the Reason is this; we are to add $b - c$; but to add b would be too much by c , therefore out of the Sum $a + b$, we must take away c : which is the same in Effect, as if the c had been first taken from b , and the Difference added to a . And here *observe*, that tho' a Line is drawn over the complex Expression added, it alters not the Effect: So $a + \overline{b - c}$ is the same in Effect as $a + \overline{b} - c$; since both ways the Difference of b and c is added to a , as has been explained.

Now, to sum up these *Cases* and *Rules* in one *General Rule*, it is this: To *Add*, join the Expressions without changing their Signs; and to *Subtract*, join them, changing all the Signs of the Subtractor; or draw a Line over the Subtractor, without changing the Signs, only join the whole Expression thus united (and, as it were, made one Expression by the Line) by the Sign $-$ betwixt it and the Subtrahend. *Lastly*, If the Numbers added or subtracted are expressed by the same Letters with Coefficients, or particular Numbers prefix'd, add or subtract these Numbers, and prefix the Sum or Difference to the common Literal Part.

II. For MULTIPLICATION.

CASE I. When two Numbers are expressed by any Letters, with particular Numbers prefix'd, (or, multiplying them) then if two or more such are to be multiplied together, multiply all the Coefficients, and prefix the Product to the Product of the Literal Expressions. Thus, $4a \times 3b = 12ab$; also, $2ab \times 5ac = 10a^2bc$.

The *Reason* is obvious; for it is only a continual Multiplication, which may be done in any Order.

CASE II. If the Multiplier and Multiplicand, one or both, are complex Expressions, the Product may be expressed by the general Rule, thus: Draw a Line over the complex Terms, and join them by the general Sign of Multiplication \times . *Example*, To multiply $a + b$ by $c + d$, the Product is $\overline{a + b} \times \overline{c + d}$; or, $a + b$ by $c - d$ makes $\overline{a + b} \times \overline{c - d}$. Which would be very different Expressions, if any of the complex Terms wanted the Line (or *Vinculum*, as the Algebraists call it;) thus $a + b \times \overline{c + d}$ would be the Sum of a , and the Product of b into $\overline{c + d}$; and $\overline{a + b} \times c - d$, is the Difference of d , and the Product $\overline{a + b} \times c$.

So that we are to understand the Sign of Multiplication to refer only to the first of the simple Terms on either hand, unless two or more of them are joined by a *Vinculum*. And here too *observe*, That when several Letters stand together, with; or without the Sign of Multiplication, (whereby they also express the Product of these Letters) this is to be reckoned but one Term, with respect to the following or preceding Sign, whether of Multiplication, Addition, &c. as $ab + d$, or $\overline{a + b} \times cd$. And mind also, that all the Terms joined together by Multiplication or Division, upon the Right-hand of the Sign of Addition or Subtraction, make but one Term to which that Sign refers; as $a + \overline{bc \times c + d}$ which is not to be understood as if $a + bd$ were multiplied by $\overline{c + d}$, which then would be made $a + bc \times \overline{c + d}$; but it is the Sum of a and the Product of bc by $\overline{c + d}$.

Observe, Tho' this general way of representing the Products of complex Expressions is often convenient, yet there is another *Rule* more useful, whereby the Product is not expressed so indefinitely, but all reduced to more simple Terms without any *Vinculum*.

Another RULE for Complex Expressions.

Multiply each simple Term of the Multiplier by each simple Term of the Multiplicand, (according to the general Rule.) And if the Signs before each of the two simple Terms, multiplied together, are the same or like (*viz.* both $+$ or both $-$) prefix the Sign $+$ to the Product; but if they are unlike (*viz.* the one $+$ and the other $-$) prefix the Sign $-$. The following *Examples* sufficiently shew the Application.

Examp. 1. $a \times b + c = ab + ac.$ *Ex. 4.* $a + b \times c - d = ac - ad + bc - bd.$
2. $a + b \times c + d = ac + ad + bc + bd$ *5.* $a - b \times c - d = ac - ad - bc - bd.$
3. $a \times b - c = ab - ac.$

DEMONSTRATION.

1. Where all the Signs are $+$, as in *Examp. 1.* and *2.* the Reason is plain, and it has also been shewn in *Lemma 3. Chap. 5. Book 1.*

2. If one of the Terms is simple, or a Sum; and the other a Difference, as in *Ex. 3; 4.* then, for *Ex. 3.* to multiply $b - c$ by a , it is evident that ab is too much; for we must take only a times the Difference of b and c , or what b exceeds c ; therefore if we take ac out of ab , the Remainder is a times the Number by which b exceeds c . Or thus; Let $b = c + d$, then $ab = a \times c + d = ac + ad$ (by the 1st *Ex.*) from which take ac , there remains ad , *viz.* the Product of a into the Difference of b and c . Again, If a Difference $c - d$ is to be multiplied by a Sum $a + b$, the Reason is the same for the Multiplication of each Term of the Sum into the Difference: As in *Ex. 4.*

3. If both Multiplier and Multiplicand are Differences, as in *Ex. 5.* then having multiplied $c - d$ by a (as in *Ex. 3.*) the Product is $ac - ad$, which is a times $c - d$, or $c - d$ times a . But, if instead of $c - d$ times a , we ought to take only $c - d$ times $a - b$, therefore, reasoning as in *Ex. 3.* it's plain we have taken too much by $c - d$ times b , or $cb - db$; and this being taken from the preceding Product $ac - ad$, the Remainder is the true Product, *viz.* $ac - ad - cb + db = ac - ad - cb + db$ (by the Rules of Subtraction) which is according to the Rule.

Or we may reason thus: Instead of taking a times $c - d$, which is $ac - ad$, we ought to take only $a - b$ times $c - d$; therefore we have taken too much by the Product of $c - d$ by b , which is $cb - db$. Or also thus: Let $a = b + n$, then $a \times c - d = b + n \times c - d = b \times c - d + n \times c - d$ (by *Artic. 1.* taking $c - d$ as one single Term.) And n being the Difference of a and b ; therefore, $n \times c - d$ is the Product sought: Consequently $b \times c - d + n \times c - d$, or its Equal $a \times c - d$, exceeds it by $b \times c - d = bc - bd$; which taken from $ac - ad$, leaves $ac - ad - bc + bd$, as before.

Observe, If any of the Terms are more complex than these *Examples*, yet the Reason is the same in all; because they are nothing else but a Sum or a Difference.

III. FOR DIVISION.

ALL the Use that is to be made of the Literal Division in the following Work, requires only, that to the General Rule I add these two Observations.

Chap. 8. Particular Rules of the Literal Arithmetick. 103

1. If the same Letter, or Expression whatever, is multiplied into all the Terms of both Divisor and Dividend, by putting it out of both, the Quote is thereby expressed more simply. Thus $\frac{ab}{a \times c + d} = \frac{b}{c + d}$. And mind, that if one Term of the Divisor or Dividend is multiplied into all the rest, then, in putting out that Multiplier, set 1 in the Place where it stood alone; thus, $\frac{ab + a}{ac - ad} = \frac{b + 1}{c - d}$ and $\frac{3a + 6}{3bc} = \frac{a + 2}{bc}$, for $6 = 3 \times 2$. The Reason of this will appear in the 4th Book. See Chap. 1. General Corollary 21. And if all the Letters represent Integers, so that the Quote thus represented has the proper Form of a Fraction; the Reason of this Rule will be seen in Book 2. Chap. 1. Lemma 5. Cor. 3.

2. If a Quote is expressed by the Sign \div or $)$ set betwixt Divisor and Dividend, it's to be referred only to the simple Terms on each hand, or which have a Vinculum: So $a + b \div c$ is the Sum of a , and the Quote $b \div c$. But however, many simple Terms (or such as become one Term by a Vinculum) are continuously joined by Signs of Multiplication and Division; we may explain them all in order as they stand, one after another. Thus $a \times b \div c \times d \div e$, may be understood thus, viz. a multiplied by b , and this Product divided by c , and this Quote multiplied by d , and this Product divided by e : Or thus, The Product ab divided by the Product cd , and this Quote divided by e : Or, a multiplied by the Quote $b \div c$, and this Product multiplied by the Quote $d \div e$. All which are equivalent. The Reason of which will appear from the Rules of Fractions, when we express the Quotes fractionally, thus, $a \times \frac{b}{c} \times \frac{d}{e}$; and this Method is generally most convenient, as it leaves no Sign but that of Multiplication.

The Application of this universal Notation, and it's Operations, in our Reasoning about Numbers, is made by means of these few simple and easy Principles, or

AXIOMS.

1. If to any Number another be added, and from the Sum be subtracted the Number added, the Remainder is the first Number: Or it is the same thing, if we first subtract and then add. Thus, $a + b = b + a$; or, $a - c + c = a$.

2. The same or equal Numbers added to the same or equal Numbers, make the Sums equal. So if we suppose $a = b$, then $a + d = b + d$; if $a + b = c + d$, then $a + b + n = c + d + n$; if $a = b - c$, then $a + c = b$. (For $b - c + c = b$ by the last.)

3. The same or equal Numbers subtracted from the same or equal Numbers, make the Remainders equal. So if $a = b$, then $a - d = b - d$; if $a + b = c + b$, then $a = c$; if $a + b = c$, then $a = c - b$; if $a + b = c - b$, then $a = c - 2b$.

4. If a Number is multiplied by another, and the Product divided by the same Number, (or first divide and then multiply) the Quote (or Product) is the first Number. So $a \times n \div n = a$.

5. If the same or equal Numbers are multiplied equally, the Products are equal. So if $a = b$, then $an = bn$; if $a + b = c + d$, then $r \times a + b = c + d$, or $ra + rb = rc + rd$; If $\frac{a}{b} = q$, then $a = bq$, (for $\frac{a}{b} \times b = a$ by the last.) And this Case is the same as the Proof of Division.

6. If the same or equal Numbers are divided equally, the Quotes are equal. So if $a = b$, then $a \div d = b \div d$, or thus, $\frac{a}{d} = \frac{b}{d}$, and if $a = bq$, then $\frac{a}{b} = q$. (For $bq \div b = q$, by the 4th.)

Observe, The Truth contained in these Axioms are universal, whether the supposed Numbers are Integers or Fractions.

BOOK

B O O K II.

Of F R A C T I O N S.

C H A P. I.

Containing the GENERAL PRINCIPLES and THEORY.

WHAT a *Fraction* is, and the *Notation* of it, has been already explained. I have shewn wherein the essential Difference betwixt *Integral* and *Fractional Numbers* does consist; and have observed, that there cannot be any other Operations in *Arithmetick* but those of *Whole Numbers*; and that the Ground and Reason of different Rules for the management of Fractions lies in their relative Nature and Value. But now more particularly *observe*, That from this relative Nature it follows, that the same Quantity may be fractionally expressed under a variety of different Forms, which in most Cases requires some preparatory Work for reducing the Numbers proposed into a like Form, before the common Operations of Addition, &c. can be performed. The first thing therefore to be done, is to explain the several Distinctions of Fractions, with the general *Theory* of their Nature; and then the Reductions of them from one Expression to another.

Observe, For brevity I contract the word *Numerator* into Num^r. and the word *Denominator* into Den^r.

D E F I N I T I O N S.

From a Comparison of the Num^r to the Den^r, as a Part (taken more generally) to the Whole; Fractions are distinguished into *Proper* and *Improper*.

1. A *Proper Fraction* is that of which the Num^r is less than the Den^r, as $\frac{2}{3}$; and is called *Proper* with respect to the relative Integer, because it expresses a Quantity less than it, (as has been already explained in a *Corollary* to the Definition of a Fraction;) as if the true and proper Signification of the word Fraction were, [a Part or Quantity less than another.

2. An *Improper Fraction* is that of which the Num^r is equal to, or greater than the Den^r. as $\frac{3}{3}$, or $\frac{4}{3}$, and is called *improper* with respect to the relative Integer, because it expresses a Quantity greater than it, and is therefore not a Part of it in any sense. But taking the word *Fraction* as I have defined it, there is no such Distinction; for each Unit of the Num^r is an *Aliquot* Part of the Integer, and the Whole is a Number of such Parts: And since the applying a relative Value or Denomination to any Number, makes it a fractional

fractional Number; therefore all such are equally true and proper, according to this larger Sense: so that wherever this distinction is applied, the meaning of the word *Fraction* is restrained; or, without minding that, we may take the Terms *Proper* and *Improper*, to signify no more than a Distinction of these different Circumstances, viz. the Numerator's being less or not less than the Denominator.

Fractions are also distinguished into *Simple* and *Compound*.

3. A *Simple Fraction* is one single Fraction, referred immediately to some Integer; as $\frac{2}{3}$, or $\frac{7}{4}$ of any thing.

4. A *Compound Fraction* is a Fraction of a Fraction, consisting of two or more simple Fractions referred to one another in order, and the last referred to some Integer; as $\frac{2}{3}$ of $\frac{4}{5}$ of any thing; or $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{7}{8}$ of any thing; the Particle, *of*, being the Mark of a Compound Fraction.

SCHOLIUM. One Fraction may be either an *Aliquot* or *Aliquant* Part of another, as well as one whole Number is of another; so that a Fraction which is $\frac{2}{3}$ of $\frac{4}{5}$, is an *Aliquot* Part of $\frac{4}{5}$; but $\frac{2}{3}$ of $\frac{4}{5}$ is *Aliquant*.

5. A *Whole Number* with a Fraction annexed, is called a *Mix'd Number*, and if the Fraction is referred to an Unit of the same thing that the whole Number represents, then they are set together without any mark of Addition, that being understood; for Example $46\frac{2}{3}$ l. but if the Fraction is not referred to an Unit of the same thing, they must be separated, that the name of each thing may be distinctly apply'd, as if it were $46\text{ l.} + \frac{2}{3}$.

SCHOLIUM. In Abstract Numbers when there is no particular thing named, a *Mix'd Number* is always understood in the first sense, (i. e. the Fraction is supposed to relate to an Unit of the same thing, which the whole Number represents) and so it's written without any mark of Addition, as $24\frac{1}{2}$. Observe also, that if we suppose (as we shall immediately prove) that two Fractions express'd in different Numbers may be equivalent, then the same integral Number, with the same or equivalent Fraction, makes the same or equivalent improper Fraction.

C O R O L L A R I E S.

1. Every *Improper Fraction* is equal to some whole or mix'd Number, and particularly, if the Numerator and Denominator are equal, the Fraction is equal to 1; for then you take as many Parts as the Integer contains, that is, the whole Integer or Unity, so $\frac{4}{4} = 1$. And where the Num^r is greater than the Denom^r, the Fraction is greater than Unity, (for the Integer.) But how to reduce it, or find the equivalent whole or mix'd Number, we shall learn afterwards.

2. Every compound Fraction is equal to some simple Fraction, for that which is a Part of a Part, is certainly a Part of the Whole, and we shall see below how to find that simple Fraction.

A X I O M S.

1. The like Fractions of two equal Quantities or Numbers are equal; that is, if $A = B$ then $\frac{a}{m}$ of $A = \frac{a}{m}$ of B .

2. If two Fractions are supposed to be equal; and if also one of their similar Terms be equal, the other is so too; thus if $\frac{a}{b} = \frac{a}{d}$ then is $b = d$.

All the other common Axioms of Numbers, hold in Fractions as well as Integers.

L E M M A I.

The greater or lesser the Num.^r of a Fraction is with the same Den.^r, the greater or lesser is the Value of the Fraction. Or thus,

Two Fractions with the same Denom.^r and different Num.^{rs}, represent quantities of different Value; and the greatest Num.^r makes the greatest Fraction. *Examp.* $\frac{3}{5}$ is greater than $\frac{2}{5}$ (of the same thing :) For the Num.^r being the direct Number of things expressed, and the Denom.^r being the same, the Value of each Unit of the Num.^r is the same, and therefore the greater Num.^r makes the greater Value in the whole.

But more particularly, if the one Num.^r is Multiple of the other, that Fraction is Equi-multiple of the other; so $\frac{4}{7}$ is double of $\frac{2}{7}$ because 4 is double of 2. For the Value of each Unit of the Num.^{rs} being equal, the comparison of them is the same as if they were pure abstract Numbers.

COROLL. A Fraction is multiplied or divided by any Integer, if we multiply or divide its Num.^r, and this is a proper Multiplication or Division of the direct Number of things represented: So $\frac{3}{7} \times 2 = \frac{6}{7}$ and $\frac{6}{7} \div 2 = \frac{3}{7}$; and universally $a \times \frac{b}{c} = \frac{ab}{c}$ and $\frac{ab}{c} \div a = \frac{b}{c}$ or the $\frac{1}{a}$ part of $\frac{ab}{c} = \frac{b}{c}$.

Observe, That the Division is supposed here to be without a Remainder, for otherways a Fraction cannot be divided by dividing its Num.^r. Because the compleat Quote being a mix'd Number, it cannot be the Num.^r of a Fraction in proper terms. *Observe* also that when the Num.^r of a Fraction is 1, it is multiplied by any Number, by placing that Number instead of the 1: thus n times $\frac{1}{a}$ Part is $\frac{n}{a}$ Parts; the truth of which needs not this *Lemma*, but is comprehended in the very Nature and Idea of a Fraction. So $\frac{3}{4}$, or $3 \frac{1}{4}$ Parts, is an equivalent Expression for 3 times $\frac{1}{4}$ Part, as every Number of any kind of things signifies so many Units of that kind.

L E M M A II.

Any Fraction of any Number is equal to the Sum of the like Fractions of all the lesser Numbers of which that Whole is composed. For *Examp.* $20 = 12 + 8$. therefore $\frac{1}{4}$ of 20 ($=5$) is $= \frac{1}{4}$ of 12 ($=3$) $+ \frac{1}{4}$ of 8 ($=2$.) Also $\frac{3}{4}$ of 20 ($=15$) is $= \frac{3}{4}$ of 12 ($=9$) $+ \frac{3}{4}$ of 8 ($=6$.) Or *Universally*, If $M = A + B + C$, &c. then $\frac{1}{m}$ of M is $= \frac{1}{m}$ of $A + \frac{1}{m}$ of $B + \frac{1}{m}$ of C . Or, $\frac{n}{m}$ of $M = \frac{n}{m}$ of $A + \frac{n}{m}$ of $B + \frac{n}{m}$ of C . [How several Fractions are added together in one simple Fraction, we learn afterwards; all that is designed here, is a general Truth, concerning a number of Fractions; for whatever way they are expressed, the general Idea is the same thing.

The *Reason* of this Truth is very plain; for the Whole being nothing else but all the Parts, when you have taken the $\frac{1}{4}$ or $\frac{3}{4}$, (or generally, $\frac{1}{n}$ Part, or $\frac{n}{m}$ Parts) of each Member of the Whole, you have taken the like Part or Parts of the Whole.

SCHOL. We may also express this Truth in this manner, *viz.* If one Quantity is made up of a number of other Quantities, $A + B + C$, and another made up of as many $a + b + c$, &c. which are respectively lesser than the former, and which are each equal to the same Fraction of their Correspondents in the other, (*i. e.* a of A , and b of B , &c.) then is the Whole $a + b + c$, &c. equal to the same Fraction of the Whole $A + B + C$, &c. that is, the Sum of the like Fractions of any two or more Numbers, is the like Fraction of the Sum of these Numbers.

COROLLARIES.

1. Any Fraction of any Number is equal to that Number of times the like Fraction of 1; for *Examp.* $\frac{2}{3}$ of 2 is $= 2 \times \frac{2}{3}$ of 1 ($= \frac{4}{3}$ of 1 *Corol. Lem. 1.*) Also $\frac{2}{7}$ of 2, is $= 2 \times \frac{2}{7}$ of 1 ($= \frac{4}{7}$ of 1.) Or *Universally*, $\frac{n}{m}$ of a is a times $\frac{n}{m}$ of 1 ($= \frac{an}{m}$ of 1, by *Cor. Lem. 1.*) For $\frac{n}{m}$ of a , is the Sum of the $\frac{n}{m}$ Parts of every lesser Number, which make up a , i. e. of every Unit in a , which is $\frac{n}{m}$ of 1, as often taken as there are Units in a , or a times $\frac{n}{m}$ of 1; that is, $\frac{n}{m}$ multiplied by $a = \frac{an}{m}$ of 1. (*Cor. Lem. 1.*) [And let it be always minded, that when the Fraction of a Number is proposed which has no such Fraction in pure Numbers, we must have recourse to applicate Numbers, so that the Number proposed is conceived to represent a Quantity subdivisible so as to have the Part proposed.]

SCHOL. In this *Corollary* we have a compleat Demonstration of what we have in part supposed in Division of whole applicate Numbers, and referr'd to this Place, viz. That any Part of any Number of Things is equal to that Number of Times the like Part of one of these Things. *Examp.* That $\frac{2}{3}$ of 2 l. $= \frac{2}{3}$ of 1 l. or $\frac{2}{3}$ of $n = \frac{2n}{3}$ of 1.

2. Any Fraction referred to any Number is equal to a Fraction whose Num^r is that given Number, and its Den^r the given one, and which is referred to the given Num^r as the Whole. Thus, $\frac{2}{3}$ of 2 $= \frac{2}{3}$ of 3: or generally, $\frac{a}{n}$ of $m = \frac{am}{n}$ of a . For $\frac{a}{n}$ of m is $= \frac{am}{n}$ of 1; and $\frac{m}{n}$ of $a = \frac{am}{n}$ of 1; (*per the last.*) Consequently, $\frac{a}{n}$ of $m = \frac{m}{n}$ of a .

3. From the two last this follows again, That any Fraction of any Number is an Aliquot Fraction having the same Den^r, and referred to the Product of the Num^r and the given Number, as the Whole: Thus, $\frac{2}{3}$ of 5 $= \frac{2}{3}$ of 10. *Universally*, $\frac{a}{n}$ of $b = \frac{a}{n}$ of ab . For $\frac{a}{n}$ of $b = b$ times $\frac{a}{n}$ of 1; or, $\frac{ba}{n}$ of 1, (by the first,) and $\frac{ba}{n}$ of 1 is $= \frac{1}{n}$ of ba , (by the last;) therefore $\frac{a}{n}$ of $b = \frac{1}{n}$ of ab .

4. The Sum of two or more Fractions having the same Den^r, is equal to a Fraction of the same Den^r, whose Num^r is the Sum of the given Num^{rs}. *Examp.* $\frac{a}{n} + \frac{b}{n} + \frac{c}{n} = \frac{a+b+c}{n}$. For $\frac{1}{n}$ of $a + \frac{1}{n}$ of $b + \frac{1}{n}$ of $c = \frac{1}{n}$ of $a+b+c$; but $\frac{1}{n}$ of $a = \frac{a}{n}$ of 1. And so of the rest: Therefore, &c.

5. Here now we have a further Demonstration of that general Truth mentioned in *Schol.* after *Lem. 2.* for demonstrating Division of Whole Numbers; viz. That the Sum of the complete Quotes of any Numbers is equal to the complete Quote of the Sum of these Numbers, being all divided by the same Divisor. For $\frac{A+B+C}{n} = \frac{A}{n} + \frac{B}{n} + \frac{C}{n}$; and whatever these Quotes are, whether Whole Numbers or Mix'd, these Fractions express the Quotes of these Numbers by the Den^r n . But again, we see here this more general Truth, viz. That whatever Numbers, (Whole or Fractional) any Number (Whole or Fractional) is resolved into, the Sum of the Quotes is equal to the Quote of the Sum, being divided by the same

Integral Divisor: For whatever kind of Numbers A, B, C are, still it is true, that $\frac{1}{n}$ of $A + \frac{1}{n}$ of $B + \frac{1}{n}$ of $C = \frac{1}{n}$ of $A + B + C$; and the $\frac{1}{n}$ of any Quantity is the Quore of it divided by n . Afterwards when we learn what the meaning of dividing by a Fraction is, we shall see the same truth hold in that Case also: So that it is Universal for all Cases, whatever the Dividend and Divisor is; as you may easily make Examples of, when you have learned the Operations of Fractions.

6. Any *Aliquot* Part of one Quantity or Number whatever, Whole or Fraction, is the same Fraction of the like *Aliquot* Part of another, as the one Whole is of the other: Or thus, any two Numbers or Quantities are the same Fractions one of another as their Equimultiples. *Examp.* If $\frac{1}{3}$ of any Quantity is equal to $\frac{2}{5}$ of the $\frac{1}{2}$ of another; then the first Whole is $\frac{2}{3}$ of the other: Or if $\frac{1}{n}$ of one Quantity is $= \frac{a}{b}$ of $\frac{1}{n}$ of another, the first Quantity is $= \frac{a}{b}$ of the other: For each Whole being composed of equal Parts, they are represented thus, $A + A + A, \&c.$ and $a + a + a, \&c.$ And the Number of Parts being equal, a and A are the like *Aliquot* Parts of their Wholes; and a is the like Fraction of A , as the Sum $a + a, \&c.$ is of $A + A, \&c.$

7. If we compare any Fraction, as $\frac{n}{m}$ of any Quantity, or Number whatever A , and the like Fraction of another Number B ; then is $\frac{n}{m}$ of A equal to the same Fraction of $\frac{n}{m}$ of B , as A is of B ; which is plain from the last: for $n \times \frac{1}{m}$ of A , and $n \times \frac{1}{m}$ of B , are Equimultiples of $\frac{1}{m}$ of A , and $\frac{1}{m}$ of B : And by the last, Equimultiples, or like *Aliquot* Parts of any two Quantities are the like Fractions one of another as these Quantities are; *i. e.* $n \times \frac{1}{m}$ of A ($= \frac{n}{m}$ of A) is the same Fraction of $n \times \frac{1}{m}$ of B ($= \frac{n}{m}$ of B), as $\frac{1}{m}$ of A is of $\frac{1}{m}$ of B : Also $\frac{1}{m}$ of A , is the same Fraction of $\frac{1}{m}$ of B , as A is of B . Hence lastly, $\frac{n}{m}$ of A is the same Fraction of $\frac{n}{m}$ of B as A of B .

LEMMA III.

The Difference betwixt the like Fractions of two Quantities or Numbers whatever, is equal to the like Fractions of the Difference of these Numbers. *Examp.* $\frac{2}{3}$ of 15 ($= 10$) $- \frac{2}{3}$ of 6 ($= 4$) is $= \frac{2}{3}$ of $15 - 6$ ($= 6$;) or generally, $\frac{n}{m}$ of $A - \frac{n}{m}$ of $B = \frac{n}{m}$ of $A - B$. Or thus, if a, b are like Fractions of A, B , *viz.* $\frac{n}{m}$ Parts, then $a - b = \frac{n}{m}$ of $A - B$.

DEMONSTR. Let $A - B = d$; then is $A = d + B$: Wherefore $\frac{n}{m}$ of $A - B = \frac{n}{m}$ of d , and $\frac{n}{m}$ of $A = \frac{n}{m}$ of $d + B$; which is $= \frac{n}{m}$ of $d + \frac{n}{m}$ of B , (by last Lemma.) Consequently, $\frac{n}{m}$ of $A = \frac{n}{m}$ of $d + \frac{n}{m}$ of B ; out of each of these take $\frac{n}{m}$ of B ; then is $\frac{n}{m}$ of $A - \frac{n}{m}$ of $B = \frac{n}{m}$ of d . But $\frac{n}{m}$ of $A - B = \frac{n}{m}$ of d , (as above;) therefore $\frac{n}{m}$ of $A - \frac{n}{m}$ of $B = \frac{n}{m}$ of $A - B$.

I

COROLL.

COROLL. The Difference of two Fractions of the same Den^r is equal to a Fraction of that Den^r, whose Num^r is the Difference of the given Numbers. *Example,* $\frac{a}{n} - \frac{b}{n} = \frac{a-b}{n}$, for $\frac{1}{n}$ of $a = \frac{a}{n}$ of 1, and $\frac{1}{n}$ of $b = \frac{b}{n}$ of 1, and $\frac{1}{n}$ of $a-b = \frac{a-b}{n}$ of 1: Therefore $\frac{a}{n} - \frac{b}{n} = \frac{a-b}{n}$.

L E M M A IV.

The more (equal) Parts any Number is divided into, the smaller these Parts are; and the fewer the Number of Parts, the greater is the Part. For *Examp.* $\frac{1}{2}$ Part is greater than $\frac{1}{4}$ Part; and so of others: For it is plain, you cannot divide the Whole into more Parts without breaking the former Parts into Pieces or smaller Parts. But more particularly, if the Den^r of one Part is Multiple of the Den^r of another, then this Part is Equimultiple of the first. For *Examp.* $\frac{1}{2}$ of any thing is double of $\frac{1}{4}$, because 6 is double of 3. And it is evident that the same reason must hold in all Cases.

Or we may express it thus; If any Quantity is divided into any Number of equal Parts, and the same (or an equal) Quantity is divided into 2, 3, 4, &c. times as many Parts, then the Part of this last Division is but $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. of the Part of the former. For you cannot make 2, 3, &c. times as many Parts, otherways than by breaking or dividing each of the former Parts into 2, 3, &c. whereby they will become $\frac{1}{2}$, $\frac{1}{3}$, &c. of the Part divided; and reciprocally if the last Division is into $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. of the Number of Parts of the first, then is the Part of the last Division 2, 3, 4, &c. times as great as that of the first. *Universally,* $\frac{1}{a}$ Part of any thing is equal to r times $\frac{1}{ra}$ Part of that thing; and Reciprocally, $\frac{1}{ra}$ Part is but the $\frac{1}{r}$ Part of $\frac{1}{a}$ Part: So that if the Den^r of an *Aliquot* Fraction is the Product of two Numbers, that Fraction is equal to the Compound of two *Aliquot* Fractions whose Den^{rs} are these Numbers. Thus, if $a = rn$, then is $\frac{1}{a} = \frac{1}{r}$ of $\frac{1}{n}$.

L E M M A V.

If two Fractions have the same Num^r with different Den^{rs}, they represent Quantities of different Values, and that which has the greatest Den^r is the least Fraction. For *Examp.* $\frac{1}{4}$ is less than $\frac{1}{3}$, because it represents the same Number of lesser Parts, (*Lem. 3.*) But more particularly, if the greater Den^r is equal to 2, or 3, &c. times the lesser, that Fraction is but $\frac{1}{2}$ or $\frac{1}{3}$, &c. of the other. And Reciprocally, this Fraction is equal to 2 or 3 times the former; so $\frac{1}{3}$ is $\frac{1}{2}$ of $\frac{1}{6}$ (and $\frac{1}{6}$ is 2 times $\frac{1}{12}$) because 3 is $\frac{1}{2}$ of 6; and $\frac{1}{7}$ is $\frac{1}{4}$ of $\frac{1}{28}$, because 7 is $\frac{1}{4}$ of 28. *Universally,* $\frac{a}{rn}$ is $\frac{1}{r}$ of $\frac{a}{n}$, and $\frac{a}{n}$ is r times $\frac{a}{rn}$, because n is $\frac{1}{r}$ of rn .

The Reason is plain: Thus, $\frac{1}{n}$ of any thing is r times $\frac{1}{rn}$ of that thing, (*Lem. 4.*) i. e. $\frac{1}{n}$ of $a = r \times \frac{1}{rn}$ of a ; but $\frac{1}{n}$ of $a = \frac{a}{n}$ of 1, and $\frac{1}{rn}$ of $a = \frac{a}{rn}$ of 1, (by *Cor. 2. Lem. 2.*) therefore $\frac{a}{n}$ of 1 = r times $\frac{a}{rn}$ of 1.

Or we may see the Truth of this somewhat otherwise; thus, $\frac{1}{n}$ and $\frac{1}{rn}$ are the like *Aliquot* Parts of $\frac{a}{n}$ and $\frac{a}{rn}$, viz. the $\frac{1}{a}$ Part; (for $\frac{1}{n} = \frac{a}{n} \div a$, or $\frac{1}{a}$ of $\frac{a}{n}$, and $\frac{1}{rn} = \frac{a}{rn} \div a = \frac{1}{a}$ of $\frac{a}{rn}$; (*Corol. Lem. 1.*) But $\frac{1}{rn} = \frac{1}{r}$ of $\frac{1}{n}$, (by *Lem. 4.*) Therefore

fore that Whole of which $\frac{1}{r n}$ is the $\frac{1}{n}$ Part, (*viz.* $\frac{a}{r n}$) is also the $\frac{1}{r}$ Part of that Whole, of which $\frac{1}{n}$ is the $\frac{1}{a}$ Part, (*viz.* $\frac{a}{n}$) *i. e.* $\frac{a}{r n}$ is the $\frac{1}{r}$ of $\frac{a}{n}$, (*Cor. 5. Lem. 2*) And Reciprocally, $\frac{a}{n}$ is r times $\frac{a}{r n}$.

C O R O L L A R I E S.

1. A Fraction is multiplied or divided by any Integer, if we divide or multiply its Den^r; So $\frac{a}{r n} \times r = \frac{a}{n}$, and $\frac{a}{n} \div r = \frac{a}{r n}$; For $\frac{a}{r n} = \frac{1}{r}$ of $\frac{a}{n}$, (by this *Lemma*) But the $\frac{1}{r}$ Part of any Quantity multiplied by r , produces that Quantity; therefore $\frac{a}{r n} \times r = \frac{a}{n}$; whence again $\frac{a}{n} \div r = \frac{a}{r n}$.

And take notice, That the Division is supposed here to be without a Remainder; for otherwise the Fraction cannot be multiplied by Division of its Den^r, because the complete Quote is a mix'd Number, and so cannot be the Den^r of a Fraction.

SCHOL. As to this *Corollary*, observe, That by multiplying or dividing the Den^r of a Fraction, what we call properly the fractional Number is not multiplied or divided, for that is the Num^r; but the mix'd Value or Quantity expressed by the Fraction is multiplied or divided; so that it is still proper to say, the Fraction (*i. e.* the Quantity expressed by it) is multiplied or divided. Again, it is manifestly the same thing in effect, to increase or diminish a Number of things, keeping the same Value of each; or to increase or diminish the Value of each, keeping the same Number; for either way the Quantity or mix'd Value of the Whole is equally increased or diminished. Hence,

2. If the Num^r and Den^r of a Fraction are equally multiplied or divided by any Number, the Products or Quotes (where there is no Remainder) make an equal Fraction: Or thus; two Fractions are equal if the Num^r and Den^r of the one are Equimultiples, or like *Aliquot* Parts of the Num^r and Den^r of the other: So $\frac{r a}{r n} = \frac{a}{n}$. For by multiplying or dividing the Number, the Fraction is multiplied or divided in the Number of things directly expressed, (*Cor. Lem. 1.*) and by multiplying or dividing the Den^r by the same Number, the Fraction is contrarily as much divided or multiplied in the Value of the things expressed, (*last Cor.*) so that what the Fraction gains or loses in the one Member, it contrarily loses or gains as much in the other; and consequently it remains still the same Fraction, only in different Terms.

3. If we find a Number which will exactly divide the Num^r and Den^r of a Fraction, we can thereby reduce it to lower Terms, (*i. e.* find another Expression in lesser Numbers, which is an equal Fraction,) *viz.* by dividing the Num^r and Den^r of the given Fraction by that Number, and taking the Quotes in place of the former Num^r and Den^r. Thus, $\frac{6}{8} = \frac{3}{4}$, by dividing 6 and 8 both by 2.

SCHOL. Such Divisions will be easily discovered in many Cases; and from the Nature of Numbers we have these particular Rules for finding a Number which will divide two other Numbers, (*i. e.* the Num^r and Den^r of a Fraction) *viz.* 1. If both are even Numbers, or have in place of Units 2, 4, 6, 8, or 0; then they are both divisible by 2; so $\frac{12}{18} = \frac{2}{3}$. And after one Division by 2, if they are still even, divide again by 2, and so on as long as they are even; thus, $\frac{32}{48} = \frac{2}{3}$.

2. If any one or both of them have 5 or 0 in the first Place, then will 5 divide them both; or if they have both 0's in the first Places on the Right-hand, cut away an equal Num-

Chap. I. General Principles and Theory of Fractions. III

Number of o's from both, (which is dividing them both equally by 10 or 100, &c. according to the Number of o's cut off;) and after these are cut off, apply some of the other Divisions, if the Case admit it.

Examp. 1. $\frac{15}{240} = \frac{1}{16}$ (viz. $15 \div 5 = 3$, and $240 \div 5 = 48$.) Ex. 2. $\frac{20}{240} = \frac{1}{12}$.

Ex. 3. $\frac{400}{1600} = \frac{1}{4}$. Ex. 4. $\frac{500}{1750} = \frac{2}{7}$.

When none of these Cases occur, yet in small Numbers you will easily discover a Number which will divide both, if there is any such; and tho' 2 will divide them, yet you'll frequently find, at first, a greater Number which will divide them.

So $\frac{7}{14} = \frac{1}{2}$, $\frac{15}{30} = \frac{1}{2}$, $\frac{24}{48} = \frac{1}{2}$, $\frac{1}{4} = \frac{1}{4}$, $\frac{24}{72} = \frac{1}{3}$.

DEFINITION 1. Two Fractions are said to be *reverse* or *reciprocal* to one another, when the Num^r and Den^r of the one is the Den^r and Num^r of the other, as $\frac{2}{3}$ and $\frac{3}{2}$, or generally $\frac{a}{b}$ and $\frac{b}{a}$ are Reciprocals; and because any integral Number is made an improper Fraction by making 1 the Den^r: Therefore a whole Number has also its Reciprocal, viz. a Fraction, whose Den^r is that whole Number, and its Num^r 1. So 2 and $\frac{1}{2}$, or a and $\frac{1}{a}$ are Reciprocals.

2. Two Fractions, whereof the Terms of the one are the 2 Num^{rs}, and the Terms of the other the two Den^{rs} of other two Fractions, are called, The Alternate Fractions of these other two: Thus $\frac{A}{C}$ and $\frac{B}{D}$ are the Alternate Fractions of these two $\frac{A}{B}$, $\frac{C}{D}$, and these Alternate of those.

LEMMA VI.

If two Fractions are equal, then these Truths follow.

1. The Products made of the Numerator of each multiplied into the Denominator of the other are equal. Thus, if $\frac{a}{b} = \frac{n}{m}$ then is $am = bn$.

Examp. $\frac{2}{3} = \frac{4}{6}$, therefore, $2 \times 6 = 3 \times 4$ ($= 12$.)

DEMONSTR. Since $\frac{a}{b} = \frac{n}{m}$ multiply each by m , the Products must be equal: But $\frac{a}{b} \times m = \frac{am}{b}$ (by Cor. Lem. 1.) and $\frac{n}{m} \times m = \frac{n}{1}$ or n , (Corol. 1. Lemma 4.) that is, $\frac{am}{b} = n$. Again, multiply each of these by b , the Products are also equal, viz. $am = bn$, which also follows from the Proof of Division.

Hence we have learned a certain Rule for trying the Equality or Inequality of two Fractions.

The Reverse of this Article is also true, viz. That if the Products made of the Numerator of each Fraction, multiplied into the Denominator of the other, are equal, these Fractions are equal. Thus, if $am = bn$, then $\frac{a}{b} = \frac{n}{m}$. For, divide am and bn both by b ,

the Quotes are equal, viz. $\frac{am}{b} = n$; and again dividing both by m , the Quotes are equal viz. $\frac{a}{b} = \frac{n}{m}$ (for $\frac{am}{b} \div m = \frac{a}{b}$.)

COROL. If $\frac{a}{b} = \frac{n}{m}$, then are n, m , both less or both greater, or both equal to their Correspondents a, b ; for since $am = bn$, if n is less than a , b must be greater than m , else it's plainly impossible that bn should be $= am$; and if n is greater than a , b must be less than m . Lastly, if $n = a$, then $b = m$.

2. The *Reciprocal Fractions* are also equal; that is, if $\frac{a}{b} = \frac{n}{m}$, then is $\frac{b}{a} = \frac{m}{n}$. Ex. if $\frac{2}{3} = \frac{4}{6}$ then is $\frac{3}{2} = \frac{6}{4}$.

DEMONSTR. Because $\frac{a}{b} = \frac{n}{m}$, therefore $am = nb$, (Art. 1.) Divide each by a , and the Quotes are equal, viz. $m = \frac{bn}{a}$: then divide each of these by n , the Quotes are equal, $\frac{b}{a} = \frac{m}{n}$.

SCHOLIUM. Because the Reciprocals of two equal Fractions are equal, we may also say, that the Reciprocal of one of two equal Fractions, is a Reciprocal to the other: So if $\frac{a}{b} = \frac{n}{m}$ then are $\frac{b}{a}$ and $\frac{m}{n}$ Reciprocals, also $\frac{a}{b}$ and $\frac{n}{m}$. But then it will be convenient to distinguish betwixt immediate and remote Reciprocals. Thus $\frac{a}{b}$ and $\frac{b}{a}$ are immediate Reciprocals, and $\frac{a}{b}$, $\frac{m}{n}$, are remote Reciprocals. But when we speak of Reciprocals in general, without distinguishing, then either of the kinds may be supposed.

COROL. If the Numerator of two reciprocal Fractions are multiplied together, and also their Denominators, the Products make a Fraction equal to 1. So if $\frac{a}{b}$, $\frac{m}{n}$, are Reciprocals, then $\frac{am}{bn} = 1$, for $\frac{a}{b} = \frac{n}{m}$ and $am = bn$, (Art. 1.) hence $\frac{am}{bn} = 1$.

3. The *Alternate Fractions* are also equal: That is, if $\frac{a}{b} = \frac{n}{m}$, then also is $\frac{a}{n} = \frac{b}{m}$ and $\frac{n}{a} = \frac{m}{b}$. Exam^p. if $\frac{2}{3} = \frac{4}{6}$, then $\frac{2}{4} = \frac{3}{6}$.

DEMONSTR. Since $\frac{a}{b} = \frac{n}{m}$ then $am = bn$ (Art. 1.) divide both by n , and the Quotes are equal, viz. $\frac{am}{n} = b$; again, divide both these by m , and the Quotes are equal, $\frac{a}{n} = \frac{b}{m}$ (for $\frac{am}{n} \div m = \frac{a}{n}$ and $b \div m = \frac{b}{m}$;) And because $\frac{a}{n} = \frac{b}{m}$ therefore $\frac{n}{a} = \frac{m}{b}$. By Art. 2.

COROL. 1. If two Fractions are equivalent, as $\frac{a}{b} = \frac{n}{m}$, the Terms of the one are like Fractions of the corresponding Terms of the other: Thus n , m , are like Fractions of a , b , or a , b , like Fractions of n , m . For $n = \frac{n}{a}$ of a , and $m = \frac{m}{b}$ of b ; but $\frac{n}{a} = \frac{m}{b}$: Also $a = \frac{a}{n}$ of n , and $b = \frac{b}{m}$ of m ; but $\frac{a}{n} = \frac{b}{m}$.

2. If two Fractions are equivalent, as $\frac{a}{b} = \frac{n}{m}$, the Terms of the one are the Quotes of an equal Division of the correspondent Terms of the other, by some Number, either integral or fractional: For by Cor. 2. Lem. 2. $\frac{a}{n}$ or $\frac{a}{n}$ of 1 is $= \frac{1}{n}$ of a , and $\frac{b}{m}$ is $= \frac{1}{m}$ of b . Suppose then, that $\frac{a}{n} = d$, then, whether d is a whole Number or Fraction, it's plain, that n times d is $= a$, and consequently d is contained in a , n times, or $a \div d = n$. Again, $\frac{a}{n} = \frac{b}{m}$, therefore, $\frac{b}{m} = d$, and for the same Reason as before, d is contained m times in b , or $b \div d = m$. In the same manner, if we suppose, $\frac{n}{a} = d$, then $n \div d = a$; and because $\frac{n}{a} = \frac{m}{b}$, therefore, $\frac{m}{b} = d$, and $m \div d = b$.

Observe,

Observe, When we have learned afterwards, that the Quote of any Number (Whole, or Fraction,) divided by any other Number, is such a Fraction of the Dividend, as the Reciprocal of the Divisor expresses, (*i. e.* the $\frac{1}{a}$ Part, if the Divisor is a whole Number a , or the $\frac{b}{a}$ Parts, if the Divisor is the Fraction $\frac{a}{b}$) then it will be plain that the one of these two *Corollaries* is contained in the other, so as either of them may be deduced from the other.

L E M M A VII.

If any Number A (whole or fractional) is equal to any Fraction (proper or improper) of another Number B, then is B equal to the reciprocal Fraction of A. *Examp.* If $A = \frac{2}{3}$ of B, then is $B = \frac{3}{2}$ of A; universally, if $A = \frac{n}{m}$ of B, then is $B = \frac{m}{n}$ of A.

D E M O N S T. If $A = \frac{n}{m}$ of B, this supposes, that B being divided into m Parts, A contains n of these Parts; which infers reciprocally, that A being divided into n Parts, B contains m of such Parts.

C O R O L. Hence we have another Proof of the 2^d Article of the preceding *Lemma*, *viz.* That if two Fractions are equal, their Reciprocals are also equal: For if $A = \frac{n}{m}$ of B, then is $B = \frac{m}{n}$ of A; and if $\frac{n}{m} = \frac{c}{d}$, therefore $A = \frac{c}{d}$ of B, and $B = \frac{d}{c}$ of A. But since also $B = \frac{m}{n}$ of A, it follows that $\frac{m}{n} = \frac{d}{c}$, else B would be equal to two different Fractions of A, which is impossible.

C H A P. II.

REDUCTION of FRACTIONS.

P R O B L E M I.

TO reduce an improper Fraction to its equivalent Whole or Mix'd Number.

Rule. Divide the Num^r by the Den^r, the Quote is the Answer.

Examp. 1. $\frac{4}{4} = 1$.

Ex. 2. $\frac{24}{6} = 4$, (the Quote of 24 by 6.)

Ex. 3. $\frac{14}{4} = 3\frac{3}{4}$, or $3\frac{1}{2}$, (because $\frac{3}{4} = \frac{1}{2}$, *Cor. 2. Lem. 4.*)

D E M O N S T. The Den^r represents the relative Integer or Unit, expressing it by a Number of Parts; therefore as oft as the Num^r contains the Den^r, it's equal to so many times that Integer, (or so many Integral Units;) and what's over in the Division, makes a Fraction of the given Denominator.

P R O B L E M II.

To reduce a mix'd Number, to an equivalent improper Fraction.

Q

Rule.

Rule. Multiply the integral Number by the Den^r of the Fraction, and to the Product add the Num^r; make the Sum a Num^r to the given Den^r, and that is the Fraction sought.

Examp. $6\frac{2}{3} = \frac{20}{3}$; thus, $6 \times 3 = 18$, then $18 + 2 = 20$.

DEMONST. This is plain from the last, for it's only the Reverse of it.

PROBLEM III.

To reduce a Whole Number to an improper Fraction, having any given Denominator.

Rule. Multiply the given Number by the given Den^r, and the Product is the Num^r of the Fraction sought.

Examp. To reduce 8 to a Fraction, having 6 for its Den^r, it is $= \frac{48}{6}$.

DEMONST. This is also plain from *Probl. 2.* being the Reverse of it.

COROL. Every Whole Number is reduced to the Form of a Fraction, by making 1 the Den^r; thus, $4 = \frac{4}{1}$.

SCHOLIUM. The same or equivalent mix'd Number, *i. e.* the same integral Number, with the same or equivalent Fraction, will always make the same or equivalent improper Fraction, only in different Terms, according as the fractional Part is. And Reverse, The same or equivalent improper Fraction will always reduce to the same or equivalent mix'd Number.

Examp. $4\frac{6}{9} = \frac{42}{9}$, and $4\frac{2}{3} = \frac{14}{3}$; and because $\frac{6}{9} = \frac{2}{3}$, therefore $4\frac{6}{9} = 4\frac{2}{3}$, consequently, $\frac{42}{9} = \frac{14}{3}$; and from this it follows reverse, that $4\frac{6}{9} (= \frac{42}{9}) = 4\frac{2}{3} (= \frac{14}{3})$.

Hence we see the Demonstration of a Truth proposed in *Schol. 2.* to Division of whole Numbers, *viz.* That the same Quote will always be expressed by the same Fraction; *i. e.* That if two Numbers are proposed to be divided by other two, if the integral Quotes are the same, when there is no Remainder; and when there is a Remainder, if the fractional Parts are also equivalent, then the Quotes taken fractionally (*i. e.* by setting the Dividend as Num^r over the Divisor) will always be equal; and if the integral or mix'd Quotes are unequal, so will these fractional ones be: And, in fine, whatever Part or Parts the lesser mix'd Quote is of the greater, the same will the equivalent fractional Quotes be. So that in the Comparison of one Quote to another, it's the same to all Intents and Purposes to express them fractionally by the Dividends and Divisors, or to reduce (*i. e.* divide) and express them directly and properly. But the Use and Conveniency of this way of expressing Quotes, we shall learn more particularly afterwards.

PROBLEM IV.

To reduce a compound Fraction to an equivalent simple Fraction.

Rule. Multiply all the Num^{rs} continually, the last Product is the Num^r sought; and multiply all the Den^{rs}, the last Product is the Den^r sought.

Examp. 1. $\frac{2}{3}$ of $\frac{5}{7} = \frac{10}{21}$, (for $2 \times 4 = 8$, and $3 \times 7 = 21$.)

Examp. 2. $\frac{2}{5}$ of $\frac{5}{7}$ of $\frac{8}{9} = \frac{80}{315}$, (for $2 \times 5 \times 8 = 80$, and $5 \times 7 \times 9 = 315$) $= \frac{16}{63}$.

DEMONST. 1. If the compound Fraction consists of two Parts, as $\frac{a}{n}$ of $\frac{c}{m}$; the Reason of the Rule is this: Since $\frac{a}{n}$ Parts of any thing is $= a$ times $\frac{1}{n}$ Part, (*Cor. Lem. 1.*)

or

or also $= \frac{1}{n}$ Part of a times that thing, (*Cor. 2. Lem. 2.*) Thence it's plain, that if we take first a times $\frac{c}{m}$, which is $= \frac{ac}{m}$, and of this take $\frac{1}{n}$ Part, which is $= \frac{ac}{nm}$; or first take $\frac{1}{n}$ Part of $\frac{c}{m}$, which is $= \frac{c}{nm}$, and then a times this, which is $\frac{ac}{nm}$: We have either way taken $\frac{a}{n}$ Parts of $\frac{c}{m}$, which gives the simple Fraction according to the Rule.

2. The same Reasoning holds if the compound Fraction has three or more Members; for the two first being reduced to one, that one and the third make the same Case as that of two Numbers; which being reduced, gives the simple Fraction equivalent to the Compound of three given ones, (which will be plainly according to the Rule, *viz.* the continual Product of Numrs and Denrs) and so on for four or more Members.

SCHOLIUMS.

1. It's no matter whether the Members of a compound Fraction be Proper or Improper, the Reduction is done the same way, and for the same general Reason, wherein there is no Regard had to the Distinction of *Proper* and *Improper*.

But this is to be observed, that if all the Members are *Proper Fractions*, their equivalent simple Fractions will necessarily be *Proper*; and if they are all *Improper*, it's *Improper*: But if some of them are *Proper* and others *Improper*, the simple ones will in some Cases be *Proper* and in some *Improper*, according as the Value of the *Proper* and *Improper* Members happen to be. But it is not to be known what it will be, otherwise than by applying the Rule, and actually finding the simple Fraction sought. So here,

$$\frac{2}{3} \text{ of } \frac{4}{5} \text{ of } \frac{7}{8} = \frac{56}{120}; \text{ but, } \frac{2}{3} \text{ of } \frac{2}{4} \text{ of } \frac{7}{8} = \frac{7}{12}.$$

2. Fractions which are referred to a Number greater than Unity, as $\frac{2}{3}$ of 3, may be also considered as compound fractional Expressions (by putting the whole Number in form of a Fraction, as $\frac{2}{3}$ of $\frac{3}{1}$, reducible to a Fraction of an Unit (of the same things) by the same Rule, (and for the same Reasons as before;) where it's plain we have nothing to do but multiply the Numr of the Fraction by the given whole Number, and apply that Product to the given Denr; so $\frac{2}{3}$ of 3 $= \frac{6}{3}$, and $\frac{3}{4}$ of 2 $= \frac{6}{4}$ ($= \frac{3}{2}$). But the more original Reason for this Case, we have already learn'd in *Cor. 2. Lem. 2.* Observe also that here, as in the other kind, the simple Fraction will in some Cases be *Proper*, and in some *improper*, even tho' the given Fraction is *Proper*; but must always be *Improper*, if the given Fraction is so. Again, We may have a Fraction referred to a mix'd Number, as $\frac{3}{4}$ of $5\frac{2}{3}$, and the Reduction to a simple Form is plainly this; Reduce the mix'd Number by *Problem 2*, and then apply the present *Problem*, thus, $5\frac{2}{3} = \frac{17}{3}$, and then $\frac{3}{4}$ of $\frac{17}{3}$ $= \frac{17}{4}$ ($= 3\frac{5}{4}$, *Prob. 1.*)

3. Some Authors propose as a kind of compound Fractions, such Expressions wherein

the Numr and Denr are themselves Fractions pure or mixed; as these, $\frac{\frac{3}{4}}{\frac{2}{5}}$ or $\frac{9\frac{5}{7}}{13\frac{1}{2}}$

or $\frac{3}{5\frac{2}{3}}$. But, in my Opinion, we cannot call any of these a Fraction with any Propriety; for they express not a certain Number of determinate Parts, which is the true and proper Notion of a Fraction. They are, indeed, reducible to an equivalent Expression in the natural Form of a Fraction; but that does not make them Fractions in the proper Notion, more than a Number of Shillings can be said to be an Expression of *Pence*, because it's reducible to such an Expression, (*i. e.* because a Number of *Pence* can be assigned

signed equal to the given Number of Shillings.) For this Reason I would never consider these Expressions as Fractions, but only as a manner of signifying that the one is to be divided by the other; or at most, as an indefinite way of expressing the Quote of the upper Number divided by the under: The finding of which Quote (or the Reduction, if you please to call it so) must therefore be learn'd from the Division of Fractions. And upon the Division of Fractions does also depend another *Problem*, which some Authors bring in among *Reductions*, viz. To find of what Number any one given Number (Whole, or Fraction, or Mix'd,) is another given Fraction. *Examp.* To find of what Number $\frac{2}{3}$ is the $\frac{2}{9}$; or to find of what Number 6 is the $\frac{2}{3}$: But these we must leave to the Rule of Division.

4. The preceding *Rule* is general, and finds the true simple Fraction required in all Cases, as has been demonstrated; and that simple Fraction may be again reduced to lower Terms, in the manner shewn in *Cor. 3. Lem. 5.* But you may more easily, in many Cases, find the simple Fraction required in lower Terms, at the first, than the General Rule gives; by this Method: First, see if the given simple Fractions can be expressed lower by the Method of *Cor. 2. Lem. 5.* and use these new Expressions in place of the former, which must certainly give the true Fraction sought; because equal Fractions are the same Fractions, only differently expressed. *Examp. 1.* $\frac{2}{3}$ of $\frac{3}{7}$ ($= \frac{20}{42}$) is the same as $\frac{2}{3}$ of $\frac{3}{7}$ ($= \frac{2}{7}$) because $\frac{2}{3} = \frac{2}{3}$. But, 2^{ly}, When you cannot reduce the given Fractions, or after you have done it, proceed thus; viz. Compare the several Num^{rs} and Den^{rs} together, and if the Num^r of one Fraction and the Den^r of another are divisible by the same Number, (which may sometimes be the lesser of these two Numbers themselves) take the Quotes and put in the Places of the Numbers divided; and do this with as many as you can; and then apply the General Rule, which will give the Fraction sought in lower Terms. The *Reason* of which is, that by this Method you have done the same in effect, as if you had found the simple Fraction by the General Rule, without such previous Work, and then divided both Num^r and Den^r by these Numbers which were made Divisors in the previous Work.

The following *Examples* will illustrate this sufficiently. I have made *Examples* only with two Members; but you can easily do the same when there are more Members: And as for such *Examples* as these, where there are but two Members, there will be no need to set down the Effect of the preparatory Work, but the Answer of the Question all at once, the intermediate Steps being easily done without writing. The finding the simple Fraction in the smallest Numbers possible, depends upon the next *Problem*, which you are to apply to the Fraction found by the preceding Rule.

$$\text{Examp. 2. } \frac{2}{3} \text{ of } \frac{3}{7} = \frac{2}{3} \text{ of } \frac{3}{7} = \frac{2}{7} \text{ of } \frac{3}{7} = \frac{2}{7}.$$

$$\text{Ex. 3. } \frac{2}{3} \text{ of } \frac{3}{7} = \frac{2}{3} \text{ of } \frac{3}{7} = \frac{2}{7}.$$

$$\text{Ex. 4. } \frac{2}{7} \text{ of } \frac{14}{19} = \frac{2}{7} \text{ of } \frac{14}{19} = \frac{4}{19}.$$

$$\text{Ex. 5. } \frac{2}{7} \text{ of } \frac{14}{19} = \frac{2}{7} \text{ of } \frac{14}{19} = \frac{4}{19}.$$

$$\text{Ex. 6. } \frac{2}{3} \text{ of } \frac{3}{7} = \frac{2}{3} \text{ of } \frac{3}{7} = \frac{2}{7}.$$

$$\text{Ex. 7. } \frac{2}{3} \text{ of } \frac{3}{7} = \frac{2}{3} \text{ of } \frac{3}{7} = \frac{2}{7}.$$

$$\text{Ex. 8. } \frac{2}{3} \text{ of } \frac{3}{7} = \frac{2}{3} \text{ of } \frac{3}{7} = \frac{2}{7}.$$

$$\text{Ex. 9. } \frac{2}{3} \text{ of } \frac{3}{7} = \frac{2}{3} \text{ of } \frac{3}{7} = \frac{2}{7}.$$

C O R O L L A R I E S.

1. In whatever Order the Members of a Compound Fraction are taken, it is still equal: So $\frac{2}{3}$ of $\frac{3}{7} = \frac{3}{7}$ of $\frac{2}{3}$; and $\frac{2}{3}$ of $\frac{3}{7}$ of $\frac{7}{9} = \frac{7}{9}$ of $\frac{2}{3}$ of $\frac{3}{7}$.

Or also exchanging the Num.^{rs} and Den.^{rs} of any two of the Members, it is still equal: So $\frac{2}{7}$ of $\frac{3}{7} = \frac{3}{7}$ of $\frac{2}{7}$. In short, the same Number of Simple Fractions make an equal Compound one, if the Num^{rs} of the Simples in each, and also the Den^{rs} are the same Numbers, tho' in such Order as not to make the same simple Fractions. The Reason is, because the

the Simple Fraction to which each of these Compounds is reduced, will be the same, being produced by the same Numbers.

2. Hence we learn how to dissolve a Fraction (if possible) into two or more component Parts; *i. e.* to reduce a Simple Fraction to a Compound one: Thus, if we can discover two, or three, or more Numbers, which multiplied together will produce a Number equal to the Num^r of the given Fraction, and as many which will produce a Number equal to the Den^r; then, of these Numbers we may make as many Simple Fractions, which, connected as the Members of one Compound Fraction, will be equal to that Simple Fraction. *Examp. 1.* $\frac{6}{20} = \frac{2}{5}$ of $\frac{3}{4}$. *Examp. 2.* $\frac{3}{18} = \frac{1}{6}$ of $\frac{1}{3}$. *Examp. 3.* $\frac{2}{4} = \frac{1}{2}$ of $\frac{1}{2}$, or $\frac{1}{2}$ of $\frac{1}{4}$ of $\frac{2}{1}$. *Examp. 4.* $\frac{3}{42} = \frac{1}{14}$ of $\frac{3}{7}$ of $\frac{1}{2}$. And this Resolution does not depend upon the Simple Fraction's being reducible to lower Terms; for this Fraction $\frac{8}{15}$, which is not reducible to lower Terms, is yet equal to $\frac{2}{3}$ of $\frac{4}{5}$. In short, as many Numbers as there are which will produce the Den^r, the Fraction is reducible to a Compound having as many Members, whereof these Numbers are the Den^{rs}; and tho' the Num^r is the Product of no Numbers but 1 and itself, yet that will afford as many Num^{rs} for the Members of the Compound Fraction, as in *Examp. 2*, and 4.

DEFINITION. As one of two equivalent Fractions must be in lesser Numbers than the other, (by *Lem. 6. Cor. to Part 1.*) So that one which is expressed by the lesser Numbers, is said to be in *lower Terms* than the other: And a Fraction is said to be in its *least* or *lowest Terms*, when there cannot be another equal to it expressed in smaller Numbers: So $\frac{2}{3} = \frac{4}{6}$, and $\frac{2}{3}$ is in the lowest Terms.

P R O B L E M V.

To reduce a Fraction to its lowest Terms; *i. e.* to find an equivalent Fraction expressed in the least Numbers possible.

We have already in *Lem. 5. Cor. 3.* learnt how a Fraction may be reduced to lower Terms, by finding a Number (if there is any such) which will exactly divide both its Num^r and Denom^r: But to reduce a Fraction to its lowest Terms, (or find if it is so already) you must take the following

RULE. Divide the greater Term by the lesser, and the Divisor by the Remainder, and the last Remainder by the preceding one, continually till nothing remains. Then by the last Remainder divide the Num^r and also the Den^r; (in which Division there will be no Remainders,) the Quotes are the Terms of the Fraction sought. So that if the last Remainder is 1, the Fraction is already in its least Terms.

Examp. 1.

$$\frac{144}{560} = \frac{9}{35}$$

Operation.

$$\begin{array}{r} 144 \overline{) 560} (3 \\ \underline{432} \\ 128 \overline{) 144} (1 \\ \underline{128} \\ 16 \overline{) 128} (8 \\ \underline{128} \\ 000 \end{array}$$

Then $144 \div 16 = 9$, and
 $560 \div 16 = 35$.

Examp. 2.

$\frac{7}{27}$ irreducible;

for

$$\begin{array}{r} 7 \overline{) 27} (3 \\ \underline{21} \\ 6 \overline{) 7} (1 \\ \underline{6} \\ 1 \end{array}$$

DEMONST. There are three things to be here demonstrated, *viz.* 1. That the last Remainder will divide the Num^r and Den^r exactly, (or without a Remainder.) 2. That it is the greatest Number that will do so. 3. That the Quotes make the least equivalent Fraction.

For the first, I must premise these Truths, *viz.* 1. If any Number does measure (or divide without a Remainder) each of two or more Numbers, it will also measure their Sum; for it is contained in the Sum precisely as oft as the Sum of the times it is contained in each of the Parts, (*Lem. 2. in Division of Whole Numbers.*) Therefore the Number which measures another, will also measure all the Multiples of that other. 2. Every Dividend is the Sum of the Remainder, and that Multiple of the Divisor produced by the Integral Quote, (by the Proof of *Division.*) Hence, 3. If the Remainder of any Division measure the Divisor, it will also measure the Dividend; for it measures the two Parts of the Dividend, *viz.* the Remainder itself, and that Multiple of the Divisor produced by the Integral Quote.

From these Truths we have a clear Demonstration of the first thing proposed: For in the Operation, every Divisor and Dividend (upward from the last) is the Remainder and Divisor of the last Division: Wherefore since the last Remainder exactly divides the last Divisor, it will also measure the last Dividend; but these being the Remainder and Divisor of the preceding Division, it must also measure the preceding Dividend; and for the same Reason, the Dividend preceding that; and so on it must measure every Divisor and Dividend to the first, which are the Terms of the given Fraction; the thing to be proved.

For the second *Article.* The last Remainder is the greatest Number that will measure the Num^r and Den^r. In order to prove this, consider, That if a Number measures the Sum of two Numbers, and also any one of them, it must measure the other; for the Sum and one Part being Multiples of that Number, so is the other Part, (*Corol. 3. Lem. 2. in Division of Whole Numbers*) and every Number measures itself and its Multiples. But that Number which measures the Divisor, measures any Multiple of it, *viz.* that Multiple produced by the Integral Quote, which is one Part of the Dividend; and if the same Number also measure the Dividend, it must measure the Remainder, which is the other Part of the Dividend. Now then if the last Remainder is not the greatest Number that measures the Num^r and Den^r of the given Fraction; suppose another greater will do it: Then, by what is now shewn, that other will also measure the first Remainder, (which is the second Divisor;) and because the first Divisor (which this supposed Number measures) is the second Dividend, it will also measure the second Remainder; and so on every succeeding Remainder: Consequently it will measure the last Remainder, which is absurd; for this Number is supposed to be greater than the last Remainder: Wherefore the last Remainder is the greatest Number which measures both the Numerator and Denominator.

For the last *Article*, *viz.* That the Quotes make the Equivalent Fraction in lowest Terms: Let the given Fraction be expressed $\frac{A}{B}$, and any Fraction in lower Terms be $\frac{a}{b}$; These Terms a and b are Quotes of an equal Division of A , B , (by *Cor. 2. Lem. 6.*) But the greater the Divisor is, the lesser is the Quote. Therefore the greatest Number which measures A and B , makes the least Quotes, and consequently the least Terms of an Equivalent Fraction.

COROL. If a Fraction is not in its least Terms, the Terms of it are Equimultiples of its least Terms; and these like *Aliquot* Parts of those. Hence again, All Equivalent Fractions in different Terms are different Multiples of the least Terms.

P R O B L E M VI.

To reduce two or more Fractions to one Denominator; i. e. to find as many Equivalent Fractions having all the same Denominator.

RULE. Multiply all the Denrs continually into one another, the Product is the common Denr sought. Then multiply each of the given Numrs into the Denrs of all the other given Fractions continually; the Product is the Numr of the Fraction sought, equivalent to the Fraction whose Numr was multiplied.

Examp. 1. $\frac{2}{3}, \frac{5}{7} = \frac{14}{21}, \frac{10}{21}$. Thus, $3 \times 7 = 21$, the common Denr: Then $2 \times 7 = 14$, which makes $\frac{14}{21} = \frac{2}{3}$. And $3 \times 5 = 15$ makes $\frac{15}{21} = \frac{5}{7}$.

Examp. 2. $\frac{5}{6}, \frac{11}{15}, \frac{8}{13} = \frac{975}{1755}, \frac{132}{1755}, \frac{1080}{1755}$. Thus, $9 \times 15 \times 13 = 1755$, the common Denr. $5 \times 15 \times 13 = 975$, the first Numr. $11 \times 9 \times 13 = 1421$, the second Numr. $8 \times 15 \times 9 = 1080$, the third Numr.

DEMONSTR. The Numr and Denr of each Fraction is equally multiplied, viz. by the Denrs of all the other Fractions; consequently the Fractions produced are equivalent, by Lem. 4. Cor. 2.

SCHOLIUM. If it is proposed to reduce any Number of Fractions to as many equivalent Fractions in the lowest Terms that can be with a common Denr; it is plain, that having reduced them first according to the preceding Rule, if we can find the greatest Number that will measure the common Denr and all the new Numrs, these being divided by it, the Quotes will make the Fractions sought. But the Demonstration of the Rule for finding that greatest Number must be referred to another Place. To which I shall therefore refer this Part of the Problem; and here only observe, that tho' the given Fractions are in their lowest Terms, yet being reduced to a common Denr by the present Problem, the new Fractions will not always be in their lowest Terms that admit of a common Denr. Examp. $\frac{5}{6}, \frac{11}{15}$, are both in their lowest Terms; and being reduced, they are $\frac{75}{135}, \frac{117}{135}$, which are again reducible to these, $\frac{5}{9}, \frac{11}{15}$; for 3 measures 75 by 25, (i. e. $3 \times 25 = 75$) and 117 by 39, and 135 by 45.

COROL. Hence we have another Demonstration of Article 1. Lemma 5. viz. That two Fractions are equal when the Products are equal which are made of the Numr of each multiplied into the other's Denr: So $\frac{2}{3} = \frac{6}{9}$, because $2 \times 9 = 3 \times 6$. For when the two Fractions are reduced to one common Denr, by the preceding Rule, these Products are the new Numrs; and it is certain, that when two Fractions are reduced to a common Denr, if the new Numrs are also equal, these new Fractions are equal, and consequently so are the Fractions to which they are equal; so in the preceding Example, $\frac{2}{3}$ and $\frac{6}{9}$ being reduced, are each $= \frac{10}{15}$; therefore $\frac{2}{3}$ and $\frac{6}{9}$, which are each equal to the same, must also be equal to one another.

P R O B L E M VII.

To reduce a Fraction to an Equivalent one of any other given Denr (if possible;) i. e. to find a Numr which with that given Denomr will make an Equivalent Fraction.

RULE.

RULE. Multiply the given Denom^r by the Num^r of the Fraction, and divide the Product by its Den^r; the Quote (if there is no Remainder) is the Num^r sought.

Examp. To reduce $\frac{3}{4}$ to an Equivalent Fraction, having for its Den^r 12. It is $\frac{9}{12}$. Thus, $3 \times 12 = 36$, and $36 \div 4 = 9$, the Num^r sought.

DEMONSTR. This follows from *Corol.* to the last. For let the given Fraction be $\frac{a}{b}$, if the Den^r to which it is to be reduced be d , suppose the Num^r sought is c . And because $\frac{a}{b} = \frac{c}{d}$ by supposition, then $ad = bc$; therefore dividing both by b , it is $\frac{a d}{b} = c$, according to the Rule.

SCHOLIUM. If the Division has a Remainder, the *Problem* is plainly impossible; yet the given Fraction is equal to the Sum of two Fractions, one of which has the given Den^r, and its Num^r is the Integral Quote of the Dividend directed, by the preceding Rule; and the other has for its Num^r the Remainder of the Division, and the Den^r is the Product of the given Den^r and the Den^r of the Fraction reduced. For *Examp.* if $\frac{4}{7}$ is proposed to be reduced to the Den^r 5, I take $4 \times 5 = 20$; then $20 \div 7 = 2$, and 6 remains. Whence I conclude, that $\frac{4}{7} = \frac{2}{5} + \frac{6}{35}$. Universally, Let it be proposed to reduce $\frac{a}{n}$ to the Den^r m . And let $n \mid am = q$, and r remaining; then the *Problem* is impossible. But I say, that $\frac{a}{n} = \frac{q}{m} + \frac{r}{nm}$.

DEMONSTR. Since $\frac{am}{n} = q + \frac{r}{n}$, then dividing both by m , it is $\frac{a}{n} = \frac{q}{m} + \frac{r}{nm}$, by *Lemma 2.* For $\frac{am}{n}$ expressing the Sum of $q + \frac{r}{n}$ the m Part of $\frac{am}{n}$, which is the Sum, is = the Sum of the m Parts of q and $\frac{r}{n}$, i. e. $\frac{q}{m} + \frac{r}{nm}$.

PROBLEM VIII.

To reduce a Fraction to an Equivalent one, having a given Num^r, (if possible.)

RULE. Multiply the given Num^r by the Den^r of the given Fraction, and divide the Product by its Num^r, the Quote (if there is no Remainder) is the Correspondent Den^r sought.

Examp. To reduce $\frac{4}{5}$ to a Fraction having 18 for its Num^r; it is done thus, $18 \times 5 = 90$, and $90 \div 4 = 22\frac{1}{2}$. So the Fraction sought is $\frac{18}{22\frac{1}{2}}$. Universally; To reduce $\frac{a}{n}$ to the Num^r c , take $cn \div a = m$, then is $\frac{a}{n} = \frac{c}{m}$.

DEMONSTR. By reverfing the given Fraction, and taking the given Num^r as a Den^r, it becomes the same Case with the preceding *Problem*; and it has been shewn, that if two Fractions are equal, they are so when reverfed. But we may argue for this the same way as in that *Problem*: Thus, if $\frac{a}{n} = \frac{c}{m}$, then $am = cn$, (*Lem. 6.*) and $m = \frac{cn}{a}$.

SCHOLIUM. If there is a Remainder, the *Problem* is impossible; yet we can find two Fractions, the one of which has the given Num^r, and whose Difference is equal to the given Fraction. For which, this is the Rule; viz. Having multiplied the given Num^r into

into the Den^r of the Fraction, and divided the Product by its Num^r, take the Integral Quote as a Den^r to the given Num^r. And if from this Fraction you subtract another, whose Num^r is the Remainder, and the Den^r is the Product of the Den^{rs}, (*viz.* of the given Fraction and that last found) this Difference is equal to the given Fraction.

Examp. If it's proposed to reduce $\frac{5}{7}$ to a Fraction whose Num^r is 9; work thus, $7 \times 9 = 63$. Then $63 \div 5 = 12$, and 3 remains. And then I say, $\frac{5}{7} = \frac{9}{12} - \frac{3}{84}$, (84 being $= 7 \times 12$.) *Universally*, If it's proposed to reduce $\frac{a}{b}$ to the Num^r n ; and if $\frac{b^n}{a} = q$, with r remaining, then $\frac{a}{b}$ is not reducible to such a Num^r. But I say, $\frac{a}{b} = \frac{n}{q} - \frac{r}{bq}$.

DEMONSTR. Since $\frac{b^n}{a} = q$, and r remaining, then is $bn = aq + r$, (by the Proof of Division.) Hence dividing equally by b , it is $n = \frac{aq + r}{b}$. And again dividing by q ,

it is, $\frac{n}{q} = \frac{aq + r}{bq} = \frac{aq}{bq} + \frac{r}{bq}$, (Lem. 2.) But $\frac{aq}{bq} = \frac{a}{b}$, (Carol. 2. Lem. 5.) Wherefore $\frac{n}{q} = \frac{a}{b} + \frac{r}{bq}$. Hence lastly, by equal Subtraction, $\frac{n}{q} - \frac{r}{bq} = \frac{a}{b}$. According to the Rule.

Observe, The preceding Problems relate all to *Abstract Fractions*, *i. e.* the Fraction reduced, and that to which it is reduced, are supposed to have the same absolute Denomination, or all to be applied to the same Integer; therefore there is none mentioned. The following Problems concern Fractions as they are specially *Applicate*.

PROBLEM IX.

To reduce a Fraction of an Unit of a higher Value, to an Equivalent Fraction of an Unit of a lower Value, these Units having a known Relation to one another, i. e. the lesser being equal to a certain known aliquot Part of the other.

RULE. Take the Reciprocal of the Fraction which expresses what Part or Parts the lower Unit is of the higher, and making that with the given Fraction (of the higher) the two Members of a Compound Fraction, reduce it to a Simple, [by Prob. 4.] *i. e.* multiply the two Num^{rs} together and the two Den^{rs}, the Products make the Fraction sought. And *observe*, if the lower is an *Aliquot* Part of the higher, we have no more to do but multiply the Num^r of the given Fraction of the higher by the Den^r of that Part.

Examp. 1. To reduce $\frac{2}{3}$ of 1 *l.* to a Fraction of 1 *sb.* it is $\frac{40}{1}$ of 1 *sb.* for 1 *sb.* is $\frac{1}{20}$ Part of 1 *l.* and 1 *l.* is 20 *sb.* Therefore $\frac{2}{3}$ of 1 *l.* is $\frac{2}{3}$ of 20 *sb.* by the Rule.

Examp. 2. To reduce $\frac{4}{5}$ of 1 *l.* to a Fraction of 1 Merk, it is $\frac{12}{1}$ of 1 Merk, Thus, 1 Merk is $\frac{1}{5}$ of 1 *l.* Therefore 1 *l.* is 5 of 1 Merk, (Lem. 7.) So that $\frac{4}{5}$ of 1 *l.* is $= \frac{4}{5}$ of 5 of 1 Merk, which, according to the Rule, makes $\frac{12}{1}$, the Fraction sought.

DEMONSTR. In the preceding Examples, I have made the Reason obvious. But to demonstrate it more *Universally*; let it be proposed to reduce $\frac{a}{b}$ of a higher Unit to a Fraction of a lower, which is $\frac{n}{m}$ of the higher. I say it is $\frac{a}{b}$ of $\frac{m}{n}$ of the lower: for since the lower is $\frac{n}{m}$ of the higher, this must be $\frac{m}{n}$ of the other, (Lem. 7.) Therefore $\frac{a}{b}$ of the higher is $= \frac{a}{b}$ of $\frac{m}{n}$ ($= \frac{am}{bn}$) of the lower, according to the Rule. And if

the lower Unit is an *Aliquot* Part of the higher; all we have to do, is to multiply the Num^r of the given Fraction (of the higher) by the Den^r of the *Aliquot* Part. So it is $\frac{a \times z}{z}$.

SCHOL. If a Compound Fraction, or a Fraction of a Number greater than Unity, is proposed, first reduce it to a Simple Fraction, and then proceed as above.

PROBLEM X.

To reduce a Fraction of a lower Unit to a higher, (the lower having a known Relation to the higher.)

RULE. Make a Compound Fraction of the given Fraction (of the lower,) and that Fraction which expresses what Part or Parts the lower is of the higher; and reduce this Compound to a Simple, you have the Fraction sought.

Exam. 1. To reduce $\frac{2}{3}$ of 1 *lb.* to the Fraction of 1 *l.* it is $\frac{2}{30}$ *l.* $= (\frac{2}{15})$. For 1 *lb.* being $\frac{1}{20}$ *l.* therefore $\frac{2}{3}$ *lb.* $= \frac{2}{3}$ of $\frac{1}{20}$ of 1 *l.* which is $= \frac{2}{60}$ *l.* according to the Rule.

Exam. 2. $\frac{4}{5}$ of 1 Merk $= \frac{8}{10}$ of 1 *l.* For 1 Merk $= \frac{1}{5}$ *l.* therefore $\frac{4}{5}$ of 1 Merk is $= \frac{4}{5}$ of $\frac{1}{5}$ of 1 *l.* $= \frac{4}{25}$ *l.* according to the Rule.

The Reason of this Rule is obviously the same in all Cases, and needs not be farther insisted on.

SCHOL. In either of the two last Problems, if there are any intermediate Species betwixt the two given Units; and if instead of the Relation betwixt the higher and lower, there be given the several Relations betwixt the Extremes and the Intermediate Species, then reduce the given Fraction to the first intermediate Species, and from that to the next, till you come to the Species required. Exam. $\frac{2}{3}$ of 1 *l.* reduced to the Fraction of 1 Farthing, is $\frac{1920}{1}$; which is found either all at once by knowing that 1 Farthing is $\frac{1}{960}$ of 1 *l.* or by degrees thus, $\frac{2}{3}$ *l.* $= \frac{4}{6}$ *s.* $= \frac{8}{12}$ *d.* $= \frac{1920}{1}$ farthings. By multiplying the Numerators gradually by 20, 12, and 4.

PROBLEM XI.

To express any Applicate Whole Number, simple or mixed, by a Fraction of some superiour Integer.

CASE 1. For a Simple Number, make it the Num^r, and for Den^r take the Number of the inferiour Species which is equal to 1 of the superiour; and that is the Fraction sought. So 8 *d.* is $\frac{8}{12}$ of 1 *lb.* or $\frac{2}{3}$ of 1 *l.*

CASE 2. For a mixed Number, reduce it to the lowest Species expressed in it, and make that the Num^r; and the Number of that lower Species which is equal to 1 of the given superiour Species make the Den^r, and that is the Fraction sought.

Exam. To express 12 *lb.* 8 *d.* 3 *f.* by the Fraction of a *l.* it is $\frac{611}{960}$ *l.* for the mixed Number is 611 *f.* and 1 *f.* is $\frac{1}{960}$ *l.* therefore 611 *f.* is 611 times $\frac{1}{960}$ *l.* $= \frac{611}{960}$ *l.*

PROBLEM XII.

To find the Value of a Fraction of any Unit (or other Number) of a given Name, in Integers of lower Species, (where there are any such.)

RULE. The given Fraction being (or made) a Simple Fraction, reduce it to a Fraction of the next lower Species, (by Prob. 9.) which being improper, reduce it (by

(by Prob. 1.) and the Integral Quote is the Answer in that Species, if there is no Remainder; but if there is a Remainder, it makes a Fraction of that Species; with which you are to proceed to the next Species, and reduce as before; and so on to the lowest: then the Integral Number found in each Species, with the Fraction of the lower, if there is a Remainder, make up the complete Answer. [And observe, if the Fraction of the first, or any succeeding lower Species is *Proper*, it is plain you can have no Integer of that Species; and so you must proceed, and reduce it to the next continually till you have an *Improper* Fraction: And if you never find such a Fraction, then the given Fraction is not expressible in Integers.]

Examp. 1. $\frac{2}{3}$ of 1 l. is = 13 lb. 4 d. which I find thus, $\frac{2}{3} l = \frac{40}{3} lb$ (Prob. 10.) = 13 lb. $\frac{1}{3} lb$ (Prob. 1.) and $\frac{1}{3} lb = \frac{4}{3} d$ (Prob. 10.) = 4 d. (Prob. 1.)
 The Reason of this Rule is evident of itself.

SCHOLIUMS.

1. This *Problem* supposes the given Fraction a *Proper* one; but for an *Improper*, first reduce it, and the Integral Quote is the first Part of the Value sought. Then proceed with the Remainder according to the Rule.

2. This Rule is accommodated to all Cases, whether the lower Units be *Aliquot* or *Aliquant* Parts of the higher. But because in the Cases which most commonly occur, they are *Aliquot* Parts, therefore the Operation is the more Simple; and the Rule may be expressed thus, *viz.* Reduce the Num^r of the given Fraction (as an Integer) to the next lower Species, till the Product be equal to, or greater than the Den^r; then divide by the Den^r, the Integral Quote is the Part of the Answer in that Species: Reduce the Remainder to the next Species, and divide as before (by the Den^r) and so on to the lowest Species; and you have the Answer either in a Simple Whole Number of one Species, or Mixed of different. And if there is a Remainder upon the last Species, it makes that Part of the Answer belonging to that Species a Mixed Number with a Fraction. And this in effect is the same as the preceding Rule.

Examp. 2. To find the Value of $\frac{24}{12} l$, it is 13 lb. 8 d. $2 \frac{10}{12} f$.

Operation.

$$\begin{array}{r}
 24 l. \\
 20 \\
 \hline
 35) 480 lb (13 lb. \\
 \underline{35} \\
 130 \\
 \underline{105} \\
 25 \text{ Rem.} \\
 12 \\
 \hline
 35) 300 d (8 d. \\
 \underline{280} \\
 20 \text{ Rem.} \\
 4 \\
 \hline
 35) 80 f (2 \frac{10}{35} f. \\
 \underline{70} \\
 10 \text{ Rem.}
 \end{array}$$

This way of ordering the *Operation* is distinct and easy; and it is exactly according to the *General Rule* of the *Problem*, which you will readily perceive by comparing. For this Reduction of the Remainder, and then the Division of the Product, is exactly the *Operation* whereby the Fraction made of that Remainder is reduced to the Fraction of the next Species, and that again reduced to a Whole Number.

3. If the Fraction to be valued refers to a Whole Number, greater than 1, as $\frac{2}{3}$ of 5*l.* or to a Mix'd Fraction, as $\frac{2}{3}$ of 6 $\frac{1}{2}$ *l.* let the Expression be reduced to a simple Fraction, and then find the Value. If that simple Fraction is improper, reduce it to its equivalent Whole Number, and then find the Value of the Remainders in inferior Species. For the valuing the Fraction of a Mix'd Whole Number, as $\frac{2}{3}$ of 48*l.* 14*sb.* 8*d.* it is to be done by multiplying the Mix'd Number by the Num^r of the Fraction, and dividing the Product by the Den^r; for $\frac{2}{3}$ of any kind of Quantity is the same as $\frac{2}{3}$ of 2 times that Quantity. Or Generally, $\frac{a}{b}$ of any Quantity is $\frac{1}{b}$ of *a* times that Quantity, which shews the Reason of this Rule.

C H A P. III.

ADDITION of FRACTIONS.

DEFINITION.

ADDITION of FRACTIONS, is finding a Fraction equal to all the given Fractions taken together.

PROBLEM. To add two or more Fractions into one Sum.

Rule. Reduce all the given Fractions to simple Fractions, of one Unit, and one Den^r, (if they are not so already;) then the Sum of the Num^{rs} being made a Num^r to the common Den^r, makes the fractional Sum sought, (which may be further reduced as the Case admits.)

SCHOLIUM. In the following *Examples*, I thought it superfluous to write down the Operations; but I have set down the Effect of every Step in the Work, separating them from each other by the Mark of *Equality*, shewing that what follows is equivalent to what precedes; being only the same Fractions reduced (where it was necessary) to a different State, according to the Direction of the Rule: Which therefore being compared with the Rule, all will be clear and manifest.

$$\text{Examp. 1. } \frac{2}{7} + \frac{3}{7} = \frac{5}{7}.$$

$$\text{Ex. 2. } \frac{2}{5} + \frac{3}{7} = \frac{14}{35} + \frac{15}{35} = \frac{29}{35}.$$

$$\text{Ex. 3. } \frac{2}{3} \text{ of } \frac{4}{5} + \frac{1}{15} = \frac{8}{15} + \frac{1}{15} = \frac{9}{15} = 1\frac{6}{15}.$$

$$\text{Ex. 4. } \frac{3}{4} \text{ of } \frac{2}{7} + \frac{5}{9} \text{ of } \frac{1}{2} = \frac{6}{35} + \frac{5}{18} = \frac{108}{304} + \frac{140}{304} = \frac{248}{304}.$$

In the preceding *Examples* the Integer is supposed to be the same in all the given Fractions, therefore I have named none; but in the following we shall make them different.

$$\text{Examp. 5. } \frac{3}{7} \text{ l.} + \frac{2}{7} \text{ sb.} = \frac{6}{7} \text{ sb.} + \frac{2}{7} \text{ sb.} = \frac{8}{7} \text{ sb.} = 8 \text{ : } 10 \text{ : } 1\frac{1}{7}.$$

$$\text{Ex. 6. } \frac{3}{5} \text{ l.} + \frac{4}{7} \text{ sb.} = \frac{6}{5} \text{ sb.} + \frac{4}{7} \text{ sb.} = \frac{42}{35} \text{ sb.} + \frac{20}{35} \text{ sb.} = \frac{62}{35} \text{ sb.} = 12\frac{20}{35} \text{ sb.}$$

$$\text{Ex. 7. } \frac{2}{3} \text{ l.} + \frac{3}{4} \text{ of } \frac{5}{6} \text{ sb.} = \frac{2}{3} \text{ l.} + \frac{1}{2} \text{ sb.} = \frac{6}{3} \text{ sb.} + \frac{1}{2} \text{ sb.} = 1\frac{4}{2} \text{ sb.} + \frac{4}{2} \text{ sb.} \text{ &c.}$$

Again,

Again, when there are mix'd Fractions, as $4\frac{2}{3} + 7\frac{1}{2}$, we may either reduce these to improper Fractions, and proceed by the General Rule; or, add the fractional Parts by themselves, and the integral, and then join both their Sums. Thus in the preceding Example, the Fractions added make $\frac{2}{3} = 1\frac{1}{3}$, and the Integers make 11; so the Total is $12\frac{1}{3}$. Again, take this Example to add $24\frac{3}{4}l.$ and $16l. 10\frac{2}{5}sb.$ The Sum of the two Fractions, viz. $\frac{3}{4}l.$ and $\frac{2}{5}sb.$ will be found $15sb. 4d. 3\frac{1}{5}f.$ The Sum of the whole Numbers is $40l. 10sb.$ and the Total is $41l. 5sb. 4d. 3\frac{1}{5}f.$ Observe also, that if the relative Integers of two Fractions are not of one general Nature, so as to have some relation, there can be no Addition.

DEMONSTR. It is already shewn, in Cor. 2. Lem. 2. that if several Fractions have one Denr, the Sum of their Numrs applied to that Denr, is a Fraction equal to their Sum; but without that Lemma, this Truth will appear very simply and easily thus: The given Fractions being such, or reduced to such a State, that all the Numrs represent things of the same Denomination, both absolute and relative, [i.e. of the same Species and Value in all respects,] their Sum is therefore a Number of things of the same kind, or a Number of such Parts as the common Denr expresses of the same common Integer.

CHAP. IV.

SUBTRACTION of FRACTIONS.

DEFINITION.

SUBTRACTION is the finding a Fraction equal to the Difference of two given Fractions.

PROBLEM. To subtract one Fraction from another.

Rule. Reduce them both to simple Fractions of one Unit, and one Denr, (if they are not so) then subtract the one Numr out of the other; and make the Remainder a Numr to the common Denr, and you have the fractional-Difference sought.

The Reason of this Rule is founded upon the same Principle as that of Addition, which need not be repeated. Or may also be deduced from Lemma 3. where it's shewn that $\frac{1}{n}$ of A — $\frac{1}{n}$ of B = $\frac{1}{n}$ of A — B, that is $\frac{A}{n} - \frac{B}{n} = \frac{A-B}{n}$; because $\frac{1}{n}$ of A = $\frac{A}{n}$, and $\frac{1}{n}$ of B = $\frac{B}{n}$, and $\frac{1}{n}$ of A — B = $\frac{A-B}{n}$ (Cor. 2. Lem. 2.)

Examp. 1. $\frac{3}{5} - \frac{2}{5} = \frac{1}{5}$.

Ex. 2. $\frac{3}{5} - \frac{4}{7} = \frac{21}{35} - \frac{20}{35} = \frac{1}{35}$.

Ex. 3. $\frac{3}{4} - \frac{1}{8}$ of $\frac{1}{4} = \frac{3}{4} - \frac{1}{8} = \frac{6}{8} - \frac{1}{8} = \frac{5}{8}$.

Ex. 4. $\frac{3}{4}l. - \frac{5}{8}s. = \frac{60}{4}s. - \frac{5}{8}s. = \frac{360}{4}s. - \frac{5}{8}s. = \frac{355}{8}s. = 44\frac{3}{8}s. = \&c.$

Ex. 5. $\frac{2}{3}$ of $\frac{4}{5}l. - \frac{6}{7}s. = \frac{8}{15}l. - \frac{6}{7}s. = \frac{160}{15}s. - \frac{6}{7}s. = \frac{1120}{105}s. - \frac{90}{105}s. = \frac{1030}{105}s. = 9\frac{24}{21}s. = \&c.$

When there is a Whole Number concerned, either in the Subtractor or Subtrahend, or both, the Difference may be found also by the General Rule; after reducing Whole and

and Mix'd Numbers to Improper Fractions: But such Cases may be solved easier, by reducing only the fractional Parts according to the Rule, and then subtracting Fraction from Fraction, and Whole Number from Whole Number. See *Ex. 6, 7.* below. Observing this, that where the absolute Denomination is the same, and the fractional Part of the Subtractor is greater than the Subtrahend, borrow Unity; *i. e.* add the Den^r (which represents the integral Unit) to the Num^r, and then subtract, and for that carry and add 1 to the Whole Number of the Species to which the Fraction refers: And if there is no Fraction in the Subtrahend, suppose one, whose Num^r and Den^r are equal each to the Den^r in the Subtractor, and the Fraction therefore equal to 1; and subtract from it (that is, subtract the Num^r of the Subtractor from its Den^r) and for that add 1 to the Whole Number of that Species, (*Ex. 8, 9.*) And if there is no Whole Number of that Species in the Subtrahend, or less than that to be subtracted from it, (*Ex. 12.*) you must supply it as in Subtraction of Whole Numbers. And *lastly*, mind that the Unity borrowed for the integral Part, must be repaid to the Integrals of the next Species, (*Ex. 11.*) and also the Unity which may happen to be borrowed for the Fraction of that next Species, (*Ex. 13.*) But the following *Examples* will make all clear.

Examp. 6.

Subtrahend.	$7\frac{3}{4} = 7\frac{9}{12}$
Subtractor.	$4\frac{2}{3} = 4\frac{8}{12}$
Differ.	$3\frac{1}{12}$

Examp. 7.

	$24\frac{2}{5} = 24\frac{4}{10}$
	$17\frac{5}{8} = 17\frac{25}{40}$
Diff.	$7\frac{14}{40}$

Examp. 8.

	14
	$8\frac{2}{7}$
Diff.	$5\frac{5}{7}$

Examp. 9.

	46
	$\frac{7}{12}$
Diff.	$45\frac{5}{12}$

Examp. 10.

	$22\frac{1}{2}$
	16
Diff.	$6\frac{1}{2}$

Examp. 11.

l.	fb.	d.
32	12	2
	6	$\frac{4}{12}$
Diff.	26	$\frac{8}{12}$

Examp. 12.

l.	fb.	d.
28	$14\frac{6}{7}$	9
	$6 : 10\frac{2}{7}$	$2\frac{6}{7}$
Diff.	22	$4\frac{3}{7} : 6\frac{2}{7}$

Examp. 13.

l.	fb.	d.	l.	fb.	d.
82	$09\frac{2}{7}$	00	=	82	$09\frac{2}{7}$
60	$14\frac{2}{3}$	$08\frac{2}{3}$	=	60	$14\frac{2}{3}$
Diff.				22	$12\frac{2}{21}$

Such *Examples* as the 12th and 13th may perhaps never (or very seldom) occur in Business, yet they are an useful Exercise to compleat one's Notion of the Nature of Fractions.

Subtraction of Fractions is proved by Addition; the same Way, and for the same Reason, as in Whole Numbers.

C H A P. V.

M U L T I P L I C A T I O N *of* F R A C T I O N S.

D E F I N I T I O N.

TO multiply any Number or Quantity by a Fraction, is no other thing than taking such a Part or Parts of it as that Fraction expresses.

PROBLEM. *To multiply one Fraction by another.*

Rule. Reduce both to simple Fractions (if they are not so) and then multiply the two Num^{rs} together, and the two Den^{rs}; their Products make a Fraction, which is the Product sought.

Examp. 1. $\frac{2}{3} \times \frac{5}{7} = \frac{10}{21}$. *Ex. 2.* $\frac{3}{5}$ of $\frac{2}{3} \times \frac{5}{12} = \frac{1}{7} \frac{3}{8} \frac{5}{5} (= \frac{27}{158}.)$

When either of the two Factors is a Whole Number, and the other a Fraction or Mix'd, or both being Mix'd; these Cases come also under the preceding Rule, if the Whole or Mix'd Numbers are first reduced to the Form of simple Fractions, (as in the following *Examples*.) So that when there is a Whole Number to be multiplied into a Fraction, it's plain we have no more to do but multiply the Num^r by that whole Number; so 4. by $\frac{3}{5} = \frac{12}{5}$: And the Reason of this we have also learn'd before, *Cor. Lem. 1.* for $\frac{3}{5}$ of 4. is 4 times $\frac{3}{5} = \frac{12}{5}$; so that it's no matter which of these you call the Multiplier.

Examp. 3. $24 \times \frac{3}{5} = 7 \frac{2}{5} (= 14 \frac{2}{5}).$

Ex. 4. $37 \frac{2}{3} \times 6 \frac{7}{8} = \frac{113}{3} \times \frac{55}{8} = \frac{6215}{24} (= 258 \frac{23}{24})$

Ex. 5. $64 \times 8 \frac{2}{3}$ or $\frac{5}{7} = 64 \times 8 \frac{10}{21} = 6 \frac{4}{1} \times \frac{178}{21} = \frac{12104}{21} (567 \frac{8}{21}).$

SCHOLIUM. After the Product is found by the General Rule, it may be reduced to lower Terms: But this Multiplication being nothing else than the Reduction of a compound Fraction to a simple, we may apply the Directions in *Schol. 4. Probl. 4. Reduction*, for finding the Product in lower Terms than the General Rule gives it; of which see the *Examples* there explained, which will be needless to repeat here.

DEMONST. To multiply by a Fraction signifies no more by the Definition, than to take such a Part or Parts of the Multiplicand as that Fraction expresses; *i. e.* plainly taking the given Fractions as Members of a compound Fraction, and reducing it to a Simple; which is done (see *Probl. 4. Reduction*) precisely according to the Rule here given: So to multiply $\frac{2}{3}$ by $\frac{4}{5}$ is no other thing than taking $\frac{4}{5}$ of $\frac{2}{3} = \frac{8}{15}$, (by *Probl. 4.* and this Rule.) And because it's equivalent in what Order the Members of a compound Fraction are taken, therefore it's the same which of them is called the Multiplier or Multiplicand.

For the *Proof* of this Work, it cannot have any which is a more simple or easier than itself; but it has a counter Operation in Division, as we shall explain when we come to it.

For APPLICATE NUMBERS.

Any absolute Denomination may be applied to one of the Terms; but the other must be abstract by the Definition; for it can signify only what Part or Parts of the other are to be taken, and the Product is Applicate to the same Things, which is to be further reduced as the Case requires. For *Example*, $\frac{3}{4}l. \times \frac{5}{8} = \frac{15}{32}l. = 12/b. 6d.$

If the Multiplicand is a Mix'd, Applicate, Whole Number, and the Multiplier a Fraction, then reduce the former to the lower Species; and if there is in that Term a Fraction, make the whole an improper Fraction, and then apply the Rule.

Examp. 6. To multiply $24l. 12/b. 8\frac{3}{7}d.$ by $4\frac{5}{9}$. By Reduction they are equal to $5912\frac{3}{7}d. \times 4\frac{5}{9} = \frac{41387}{7} \times \frac{41}{9} = \frac{1696867}{63}d. = 26934\frac{31}{63}d. = \&c.$

But here if the Multiplier is a Whole Number, and a small one, so that the Multiplication can easily be performed without previous Reduction, let it be done that way; beginning with the Fraction in the lowest Species. *Examp. 7.* $14l. 12s. 10\frac{2}{5}d.$ by $4 = 58l. 11s. 5\frac{2}{5}d.$ Thus $\frac{2}{5} \times 4 = \frac{8}{5} = 1\frac{3}{5}d.$ which 1 is carried to the Product of Pence. *Examp. 8.* $78l. 14s. 6d.$ by $\frac{5}{9}$: Multiply the Mix'd Number by 5, then divide the Product by 9; and this is a Question of that same kind which we have seen already in *Schol. 3. Probl. 12. Reduction.*

GENERAL SCHOLIUM.

The word Multiplication, more properly and strictly taken, signifies the encreasing of a Number by Repetition; whereas to multiply by a proper Fraction (according to the preceding Rule) does plainly find a Number less than the Multiplicand; which does therefore rather divide than multiply it. But more particularly, when the Multiplier is an Aliquot Fraction, as $\frac{1}{3}$, the Effect of this Operation is plainly nothing else but Division, viz. of the Multiplicand by 3, which finds $\frac{1}{3}$ of it. Again, If the Multiplier is a Fraction of any other kind, proper or improper, as $\frac{2}{3}$ or $\frac{7}{5}$, the Operation is mix'd, whereby the Multiplicand is first multiplied and the Product divided; for we take a certain Part of a certain Multiple of it. But this Difference is remarkable, viz. That if the Multiplier is a proper Fraction, the Division prevails, and the Number said to be multiplied is really lessened: But if it's an Improper Fraction, the Multiplication prevails, and the Multiplicand is encreased. The first is therefore in some sense more properly a Division, and the last a Multiplication; tho', according to the Definition, they are both called Multiplication; (nor does the first agree to the Definition in Division, as we shall see in the next Chapter.) Again, Take notice, that if a Whole Number and a Proper Fraction are multiplied together, the Fraction is, in a strict and proper Sense, multiplied; but the Whole Number is lessened, and is only multiplied in that Sense in which Multiplication by a Fraction is here defined. And now at last if you enquire, How the Name of Multiplication comes to be applied to a Work which really diminishes? it seems to be from this Consideration, viz. That whether the Multiplier is a Fraction or Whole Number, the Number found has the same relation to the one Factor as the other has to 1; i. e. it contains it as oft, or as many Parts of it, as the other expresses, or as it contains Unity or Parts of Unity; which is plain from the Definition. So in Whole Numbers, $a \times b$ (or the Product of a by b) contains a , b times, and so does b contain 1, b times. In Fractions the Product of a by $\frac{n}{m}$ is $\frac{n}{m}$ Parts of a , as $\frac{n}{m}$ expresses $\frac{n}{m}$ Parts of 1: And because of this general Likeness in the Effect, both are called Multiplication; which is also defined in this general manner, viz. Finding a Number which shall have the same relation (above explained) to one of the two given Numbers, as the other has to Unity: tho' the Effect of this is in some Cases really Division, and in all others, is mix'd of Multiplication and Division, taking these in their more strict and proper Sense.

C H A P.

C H A P. VI.

D I V I S I O N *of* F R A C T I O N S.

D E F I N I T I O N.

DI V I S I O N is taken here in the same Sense as already explained in Whole Numbers, *viz.* finding how oft one Fraction is contained in another.

G E N E R A L S C H O L I U M.

We see now by comparing the General Nature of Multiplication and Division by Fractions, as it appears in their Definitions, that they are the same way opposite in their Effects, as in Whole Numbers. For, let any Number A (Whole or Fraction) be multiplied by any Number B, (Whole or Fraction) and let the Product be represented by D; then D contains A, B times, or such a Fraction of A as B expresses, (according to the Nature of Multiplication by Fractions;) consequently, if we divide D by A, *i. e.* enquire how oft A (or what Fraction of it, if it's greater than D) is contained in D, the Quote must be B: And reversely, if any Number (Whole or Fraction) is contained so oft (or such a Fraction of it) in D, as B expresses, then must so many times (or such a Part of) A, as B expresses, be equal to D. Therefore, whatever be the Rule for finding how oft any Fraction is contained in another Number, this is certain, that the Quote multiplied into the Divisor (according to the Rules of Fractions) must produce the Dividend, or its equivalent; for it may arise in different Terms, as we shall presently see.

PROBLEM. *To divide one Fraction by another.*

Rule. Reduce both to simple Fractions, then take the Dividend and the Reciprocal of the Divisor, as Members of a Compound Fraction, and multiply them together; the Simple Fraction produced is the Quote sought; *i. e.* the Quote is equal to such a Fraction of the Dividend as the Reciprocal of the Divisor expresses: Or thus, (which is the same thing) Multiply the Num^r of the Dividend into the Den^r of the Divisor, then the Den^r of the Dividend into the Num^r of the Divisor; make the first Product the Num^r, and the other Den^r of a Fraction, and it's the Quote sought; which is to be further reduced according to the Circumstances and Sense of the Question.

Examp. 1. $\frac{3}{39} \div \frac{6}{13} = \frac{234}{39} = 6.$

Ex. 2. $\frac{2}{3} \div \frac{8}{9} = \frac{24}{8} = 1\frac{6}{8} = 1\frac{3}{4}.$

Ex. 3. $\frac{3}{4} \div \frac{2}{5} = \frac{15}{8}.$

Ex. 4. $\frac{2}{3} \text{ of } \frac{5}{6} \div \frac{8}{15}; \text{ or } \frac{5}{9} \div \frac{8}{15} = \frac{75}{72}.$

If there is a Fraction any way concerned in either of the Terms, *i. e.* if either of them is a whole Number, and the other a Fraction or mix'd, or both mix'd Numbers, they come both under the preceding Rule, if the Whole or Mix'd Number is first reduced to the Form of a Simple Fraction, (as in the following *Examples.*) So that when either of them is a Whole Number, we have no more to do but multiply the Num^r or Den^r of the other by it, according as that whole Number is the Dividend or Divisor.

S

Examp.

Examp. 5. $24 \overline{) 368 \frac{5}{6}}$ or $\frac{24}{1} \overline{) \frac{368 \frac{5}{6}}{1}} = 15 \frac{5}{4}.$

Ex. 6. $8 \overline{) \frac{24}{39}} = \frac{1}{13}.$

Ex. 7. $2 \frac{3}{4} \overline{) 6 \frac{5}{7}}$ or $\frac{11}{4} \overline{) \frac{42 \frac{5}{7}}{1}} = 2 \frac{3}{7}.$

Observe. The preceding *Examples* are all in abstract Numbers; and for the Management of applicate Numbers, where Fractions are concerned, I shall consider them by themselves, because of some things that require to be particularly explained as to the Sense and Meaning of Division applied in some Cases; but I shall first demonstrate the preceding General Rule.

DEMONST. The Reason of this Rule may be variously deduced thus: (1.) That is the true Quote, which, multiplied by the Divisor, produces the Dividend, (by what's shewn in the preceding *Gen. Schol.*) Now if $\frac{c}{d}$ is divided by $\frac{a}{b}$, the Quote according to the Rule is $\frac{bc}{ad}$, which multiplied by the Divisor $\frac{a}{b}$ (i. e. take $\frac{a}{b}$ of $\frac{bc}{ad}$) the Product is equal to $\frac{abc}{bad}$. But this being reduced to lower Terms, viz. by dividing both Num^r and Den^r by ab , (or ba ,) becomes equal to $\frac{c}{d}$ the given Dividend; therefore $\frac{bc}{ad}$ is the true Quote.

Or, 2. We may prove it thus: Suppose $\frac{c}{d} \div \frac{a}{b} = \frac{n}{m}$, that is, $\frac{a}{b}$ is contained in $\frac{c}{d}$ $\frac{n}{m}$ times; wherefore $\frac{n}{m}$ times $\frac{a}{b}$, or $\frac{n}{m}$ of $\frac{a}{b} = \frac{c}{d}$: But $\frac{n}{m}$ of $\frac{a}{b} = \frac{a}{b}$ of $\frac{n}{m}$, which is therefore $= \frac{c}{d}$; and hence (by *Lem. 7.*) $\frac{n}{m} = \frac{b}{a}$ of $\frac{c}{d}$, which, according to the Rule, is the Quote of $\frac{c}{d} \div \frac{a}{b}$.

Now, tho' either of these two is a strict Demonstration of the Rule, yet being deduced only from the Opposition betwixt Multiplication and Division, it will be useful to see the Reason and Invention of it more directly and immediately from the Nature of Division itself. Thus,

3. Let it be required to divide $\frac{c}{d}$ by $\frac{a}{b}$. If we first enquire how oft is a contained in $\frac{c}{d}$, the Quote is $\frac{c}{da}$ (by *Cor. 1. Lem. 5. Chap. 1.*) But because we ought to enquire, how oft $\frac{a}{b}$ or $\frac{1}{b}$ of a is contained; and this must be b times as oft, therefore multiply the last Quote $\frac{c}{da}$ by b , the true Quote is $\frac{cb}{da}$ ($= \frac{b}{a}$ of $\frac{c}{d}$,) according to the Rule.

4. We have yet a more simple View of it, thus: Suppose the Divisor and Dividend have (or are reduced to) one common Den^r, then it's evident that the Dividend contains the Divisor as oft, or as many Parts of it, as its Num^r does the other; for having one Den^r, they are in the same State with respect to one another, as Whole Numbers: So that the Num^r of the Dividend, set fractionally over the Num^r of the Divisor, expresses the true Quote. For it's evident, that $\frac{a}{x}$ contains $\frac{b}{x}$ as oft as a contains b , since these Numbers express Units of the same Value. Now, tho' there is no word in the Rule of reducing to one Den^r, yet the fractional Quote, found by the Rule, is plainly the same as that now mentioned; for the Operation is the same as that by which the Num^{rs} are found, when they are reduced to a common Den^r. Thus, if $\frac{a}{b}$ and $\frac{c}{d}$ are reduced to a common Den^r,

they

they are $\frac{ad}{bd}$ and $\frac{bc}{bd}$, and the Fraction made of their Num^{rs}; *i. e.* the Number of times $\frac{ad}{bd}$ is contained in $\frac{bc}{bd}$, is truly expressed by $\frac{bc}{ad}$, which is the Quote of $\frac{c}{d}$ divided by $\frac{a}{b}$, according to the Rule.

COROLLARIES.

If the Divisor and Dividend, being simple Fractions, (or reduced to that State) have a common Den^r, their Quote is the Quote of the Num^{rs} taken by the Rule of Division of whole Numbers. So $\frac{2}{7} \div \frac{3}{7} (\frac{2}{7} \text{ and } \frac{3}{7}) \frac{2}{3} = 2$.

II. From the General Rule this plainly follows: That the Reciprocal of any Quote will be the Quote when the Divisor and Dividend are changed. For *Example*, $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$, and $\frac{c}{d} \div \frac{a}{b} = \frac{bc}{ad}$. Or thus, if $\frac{a}{b} \div \frac{c}{d} = \frac{n}{m}$, then is $\frac{n}{m}$ of $\frac{c}{d} = \frac{a}{b}$, and $\frac{c}{d} = \frac{n}{m}$ of $\frac{a}{b}$; therefore $\frac{c}{d} \div \frac{a}{b} = \frac{m}{n}$. And the same thing being true in whole Numbers also, we learn this general Truth, *viz.* That if any two Numbers are divided, either by the other, the Quote of the one by the other, is the Reciprocal of the Quote of that other by the former.

SCHOLIUM. When the Quote is found, it may be reduced to lower Terms; but if you consider the Dividend and the Reciprocal of the Divisor, as Members of a compound Fraction, (according to the Rule) then you may apply the Directions given in *Schol. 4. Probl. 4. Reduction*, for finding the Quote in lower Terms. But the Rules for the more useful of these Cases, applied to Division, may be expressed in this manner, *viz.*

(1.) If the Num^r and Den^r of the Dividend are Multiples of the Num^r and Den^r of the Divisor, divide them, and make the first Quote Num^r and the other Den^r, and that Fraction is the Quote sought.

$$\text{Examp. } \frac{2}{3} \div \frac{4}{12} = \frac{2}{3}.$$

(2.) If the Num^r of the Dividend is Multiple of the Num^r of the Divisor, but not the Den^{rs}, then take the Quote of the Num^{rs} and multiply it into the Fraction made by setting the Den^r of the Divisor over the Den^r of the Dividend.

$$\text{Examp. } \frac{2}{3} \div \frac{4}{7} (2 \times \frac{3}{7} = \frac{6}{7}.$$

(3.) If the Den^r of the Dividend is Multiple of the Den^r of the Divisor, but not the Num^{rs}, then take the Fraction made by setting the Num^r of the Dividend over the Num^r of the Divisor, and divide it by the Quote of the Den^{rs}; *i. e.* multiply its Den^r by that Quote.

$$\text{Examp. } \frac{2}{3} \div \frac{5}{6} (\frac{2}{3} \div 2 = \frac{2}{6} = \frac{1}{3}.$$

(4.) If the Num^r of the Dividend is Multiple of the Num^r of the Divisor, and the Den^r of the Divisor Multiple of the Den^r of the Dividend, divide the Num^{rs} one by the other, also the Den^{rs}, the Product of those Quotes is the Quote sought.

$$\text{Examp. } \frac{2}{3} \div \frac{6}{13} = 6, \text{ for } 6 \div 3 = 2, \text{ and } 39 \div 13 = 3.$$

The Reason of all these Rules is contained in what is explained in the place above referred to; and were superfluous to repeat.

For APPLICATE NUMBERS.

There is the same Variety in Division of Applicate Numbers where Fractions are considered, as when all are Whole Numbers. But before we make any Examples of this kind, it will be proper that we first reflect upon the four different Senses explained in Division of Whole Numbers, and consider how they are to be applied with Fractions.

First, If the Divisor is a Whole Number, the Dividend being a Fraction either pure or mixed, the different Senses are Applicable the same way as if the Dividend were also a Whole Number. For we may enquire, 1. How oft the Divisor is contained in the Dividend. 2. What Part it is of the Dividend. Or, 3. What Number (Whole or Fraction) is contained in the Dividend as oft as the Divisor expresses. Or, 4. What is that Number which is such a Part of the Dividend as the Divisor denominates. In all which there is no matter what the Dividend is; for the Answer of the Question in the first Sense will answer it in them all; except in the second, when the Divisor is not an *Aliquot* Part of the Dividend; as has been already explained.

In the next Place, let the Divisor be a Fraction pure or mixed; whatever the Dividend is, the first two Senses are applicable without any Variation: for we may reasonably ask, 1. How oft a Fraction is contained in any Number. Or, 2. What Part it is of any Number. *Observing* this, That whether the Divisor is an Integer or Fraction, if it is not an *Aliquot* Part, yet we may ask what Fraction it is of the other. 3. We may enquire what is that Number which is contained in the Dividend such a Fraction of a time (*i. e.* of which the Dividend contains, or is equal to such a Fraction) as the Divisor expresses. It is plain, the Quote taken in the first Sense will answer this also; because the Quote and Divisor produce the Dividend. For let $\frac{c}{d} \div \frac{a}{b} = \frac{n}{m}$, which in the first Sense signifies that $\frac{a}{b}$ is contained in $\frac{c}{d}$, $\frac{n}{m}$ Parts of a time; *i. e.* that $\frac{c}{d} = \frac{n}{m}$ of $\frac{a}{b}$. But $\frac{n}{m}$ of $\frac{a}{b} = \frac{a}{b}$ of $\frac{n}{m}$, (by *Cor. 1. Probl. 4* of Reduction.) Therefore $\frac{c}{d} = \frac{a}{b}$ of $\frac{n}{m}$, *i. e.* $\frac{c}{d}$ contains $\frac{n}{m}$, $\frac{a}{b}$ Parts of a time. Wherefore $\frac{n}{m}$ is the Quote in this third Sense also. And this Question applied to a fractional Divisor, is parallel to the third Sense applied to an Integral Divisor. And to comprehend both without distinguishing, we may ask what is the Number which is contained in the Dividend so many times, or such a Fraction of a time as the Divisor expresses.

4. The fourth Sense in Whole Numbers is finding a Number which is such a Part of the Dividend as the Divisor denominates; which is in effect multiplying by its reciprocal Fraction; for it's plainly taking such a Fraction of the Dividend as the Reciprocal of the Divisor expresses. So to divide by $\frac{1}{4}$, in this Sense, is to take $\frac{1}{4}$. And to make a Question like this with a fractional Divisor, we must seek a Number which is such a Fraction of the Dividend as the Reciprocal of the Divisor expresses; which is the immediate effect of the Rule given for finding the Quote in the first Sense: So that it is plain, the Quote which answers the Question in the first Sense, does so in this Sense also; *i. e.* is a Number which is such a Fraction of the Dividend as the Reciprocal of the Divisor expresses. This therefore is a general Truth, that whatever the Divisor and Dividend be, the Quote found by the general Rule, is such a Fraction of the Dividend as the Reciprocal of the Divisor expresses.

But lastly *observe*, That if a Question is thus proposed, *viz.* to find $\frac{b}{a}$ Parts of $\frac{c}{d}$, it is directly in the Form of Multiplication, and so does not appear as a Question of Division; nor is it so in any other Sense than as the Answer of it is equal to the Number of times

times that its Reciprocal $\frac{a}{b}$ is contained in $\frac{c}{d}$: Or, as it is the Answer of a Question of Division made with the Divisor $\frac{a}{b}$ in the third Sense.

Now then, as in Division of Applicate Whole Numbers, so here in Fractions there are but two Cases: For either,

1. The Divisor and Dividend are both Applicate to the same kind of thing; and the Question is always in the first and second Sense: Therefore the Quote is an Abstract Number. And to find it, you must reduce the given Numbers to one absolute Denomination or Integer, and then apply the preceding general Rule. See *Examp.* 8, 9, 10. *below.*

2. The Dividend is Applicate, and the Divisor Abstract; then the Question can be proposed only in the third and fourth Sense. Minding that the fourth Sense is to be called Division only as it is the Answer of the Question proposed in the third Sense, with the Reciprocal of the Divisor, See *Examp.* 14, 15, 16. *below.*

Examples of CASE I.

$$\text{Ex. 8. } \frac{2}{3} l.) \frac{8}{9} l. (\frac{4}{3} = 1 \frac{1}{3}. \quad \text{Ex. 10. } 3 \frac{5}{9} l.) 18 \frac{3}{7} lb. \text{ Or, } \frac{3 \frac{2}{9} l.) \frac{1 \frac{2}{9} lb.}{7}$$

$$\text{Ex. 9. } \frac{3}{5} lb.) \frac{2}{3} l. \quad \text{Or, } \frac{6 \frac{40}{9} lb.) \frac{1 \frac{2}{9} lb.}{7} (\frac{1 \frac{1}{4} \frac{6}{8} \frac{1}{0}}{}$$

$$\text{By Reduction.} \quad \text{Ex. 11. } 3 \frac{2}{3} lb.) 26 l. \text{ Or, } \frac{1 \frac{7}{3} lb.) \frac{5 \frac{20}{3} lb.}{1} (\frac{2 \frac{600}{1} \frac{0}{7} = 152 \frac{1}{3} \frac{6}{7}.$$

$$\frac{3}{5} lb.) \frac{4 \frac{0}{3} lb.}{3} (\frac{2 \frac{0}{9} \frac{0}{9} = 22 \frac{2}{9}.$$

If either Divisor or Dividend is a mix'd Whole Number of different Species, reduce both to a Simple Number of the lowest Species; and if there is a Fraction in one or both, they must also be reduced to one Integer, and that the same to which the Integral Numbers are reduced. But this will be best done by reducing to an Improper Fraction, and then reducing this to the Integer required.

$$\text{Ex. 12. } 4 \frac{3}{8} lb.) 48 l. 18 s. 9 d. \text{ By Reduction, } \frac{2 \frac{7}{5} d.) \frac{1 \frac{1}{2} \frac{4}{5} d.}{1} (\frac{5 \frac{8}{2} \frac{7}{6} \frac{2}{5} = 212 \frac{1}{2} \frac{1}{7} \frac{4}{8}.$$

$$\text{Ex. 13. } 3 l. 4 \frac{2}{7} lb.) 38 l. 12 lb. 8 \frac{1}{3} d.$$

By Reduction.

$$64 \frac{2}{7} lb.) 9272 \frac{1}{3} d. \text{ Or, } \frac{4 \frac{50}{7} lb.) \frac{4 \frac{6}{5} \frac{3}{6} \frac{1}{1} d.}{5} \text{ Or, } \frac{4 \frac{50}{7} lb.) \frac{4 \frac{6}{60} \frac{3}{60} \frac{1}{1} lb.}{60} (\frac{3 \frac{2}{2} \frac{4}{7} \frac{5}{0} \frac{2}{0} \frac{7}{0} = 12 \frac{5}{2} \frac{1}{7} \frac{7}{0000}.$$

Examples of CASE II.

Where the Dividend is Applicate, and the Divisor Abstract; i. e. wherein is sought a Number, of which such a Fraction or Multiple as the Divisor expresses, is equal to the Dividend.

$$\text{Examp. 14. } \frac{2}{3}) \frac{8}{9} l. (\frac{4}{3} l. = 1 \frac{1}{3} l. = 1 l. 6 lb. 8 d.$$

$$\text{Examp. 15. } 4 \frac{3}{5}) 24 l. 16 s. 9 d. \text{ Or, } \frac{2 \frac{3}{5}) \frac{5 \frac{9}{1} \frac{6}{1} d.}{1} (\frac{2 \frac{9}{2} \frac{8}{3} \frac{0}{5} d. = 1295 \frac{2}{2} \frac{0}{3} d. = 5 l. 7 s. 11 \frac{2}{3} d. \quad \text{If.}$$

If the Divisor is a Whole Number, you need not always reduce the Dividend when it is a mixed Number, but carry on the Division by degrees to the lowest Species, where the Fraction is; and what is Remainder upon dividing the Integral Part of that Species, annex to the Fraction, and reduce them to an Improper Fraction, and then divide as is done in the following Example.

Examp. 16. $4 \) \ 18 \ l. \ 13 \ sb. \ 9 \frac{2}{3} \ d. \ (\ 4 \ l. \ 13 \ sb. \ 5 \ \frac{5}{12} \ d.$

GENERAL SCHOLIUM.

As we observed before upon the multiplying by a proper Fraction, that it diminishes the Number multiplied by it, contrary to the effect of an Integral Multiplier, and to the more limited Sense of the word Multiplication; so here, to divide by a proper Fraction does always quote a Number greater than the Dividend, contrary to what is done with an Integral Divisor; and contrary also to the more strict Sense of the word Division, which imports the lessening of a thing: and so the Quote will in some Cases be a Whole or Mix'd Number, tho' the Dividend is a Fraction; because one Fraction may be equal to another taken once, twice, or any Number of Times, or Parts over. Now, that the Quote must be always greater than the Dividend, is plain from the Rule. For the Divisor being a Proper Fraction, its Reciprocal is an Improper Fraction greater than Unity, by which the Dividend is multiplied, which is therefore taken more than once to make the Quote. Again more particularly, if the Divisor is an *Aliquot Fraction*, the Operation is in effect a proper Multiplication; so to divide by $\frac{1}{4}$, is no other thing than multiplying by 4. And the Divisor being a Fraction of any other kind, the Operation is mix'd of a proper *Multiplication* and *Division*, in which the Multiplication prevails, if the Divisor is a proper Fraction; but the Division prevails, if the Divisor is an improper Fraction. Now, because the Quote may be in some Cases greater than the Dividend, we may enquire, How the name *Division* came to be applied to an Operation which really increases a Number: And the Reason of this is probably, (as before we alledged in Multiplication,) the general likeness in the effect of dividing by a Whole Number and by a Fraction, *viz.* That the Quote has the same Relation to the Dividend that Unity has to the Divisor, *i. e.* that the Quote contains the same Part or Parts of the Dividend as Unity does of the Divisor. This is manifest in Whole Numbers; for if B is divided by A, the Quote is $\frac{1}{A}$ of B, and 1 is $\frac{1}{A}$ of A. In Fractions the same truth will exactly appear; Thus, let the Dividend be D, (either a Whole Number, or Fraction,) and the Divisor $\frac{a}{b}$; by the Rule, the Quote is $= \frac{b}{a}$ of D; but 1 is also $= \frac{b}{a}$ of $\frac{a}{b}$; (these being Reciprocals, *Cor. Lem. 5.*) And from hence some define *Division* in this manner, *viz.* Finding a Number which shall have the same Relation (as before explained) to the Dividend, as Unity has to the Divisor.— But we have also this Difference to observe, in the Definitions which I have applied to *Multiplication* and *Division*: That in *Division*, the Definition is the same for Fractions and Whole Numbers without changing one Word, *viz.* finding how oft the *Divisor* is contained in the *Dividend*. But in *Multiplication* there is Variety; for in Whole Numbers it is repeating the Multiplicand, and taking the Whole of it a Number of Times; but with a Fraction it is only taking a Part, or certain Parts of it. Yet this is still true, That the Operations of *Multiplication* and *Division* are reverse to one another; so that the one undoes what the other did, whereby they prove one another. And hence we have this

GENERAL COROLLARY, To the Rules of Multiplication and Division, both by Whole Numbers and Fractions; viz.

To multiply by any Number, Whole, or Fraction, and to divide by its Reciprocal, have always the same effect, or brings out the same Number; so $8 \times 3 = 8 \div \frac{1}{3}$; or generally, $A \times m = A \div \frac{1}{m}$, each being (by their proper Rules) $= Am$; or, $\frac{A}{B} \times m = \frac{A}{B} \div \frac{1}{m}$, each being $= \frac{Am}{B}$. And $A \times \frac{n}{m} = A \div \frac{m}{n}$; for each is $= \frac{An}{m}$. Also $\frac{A}{B} \times \frac{n}{m} = \frac{A}{B} \div \frac{m}{n}$, each being $= \frac{An}{Bm}$.

The last thing I have to observe, is, That some may ask, Why the finding, for Example, $\frac{2}{3}$ of a thing is not called *Division*, since $\frac{2}{3}$ is found by dividing, (*viz.* with 3.) The Answer is plainly this, Dividing by 3 is the same as multiplying by $\frac{1}{3}$. So that to find $\frac{2}{3}$ we do multiply according to the Rules of Fractions, which makes the Product a Fraction; and when this happens to be an Improper Fraction, (as it is always when the Dividend is a Whole Number greater than 3) the Division by 3 is only the Reduction of that Improper Fraction. And for the same Reason to find $\frac{2}{3}$ will also require a Division, if the Fraction produced be Improper: But to make the Comparison aright, let them consider the Difference betwixt dividing by 3 and by $\frac{1}{3}$; these cannot have the same Effect. And if dividing by 3, finds $\frac{1}{3}$ of the Dividend; dividing by $\frac{1}{3}$ does not also find $\frac{1}{3}$ of it. Indeed, the finding $\frac{1}{3}$ is in some sense *Division*; but it is not Division by $\frac{1}{3}$, but by 3. Or it is not finding how oft $\frac{1}{3}$, but how oft 3 is contained in the Dividend. In the same manner, finding $\frac{2}{3}$ is in a sense *Division*, but it is not Division by $\frac{2}{3}$, but by $\frac{3}{2}$ its Reciprocal; in the same manner as $\frac{2}{3}$ is found by dividing by 3, the Reciprocal of $\frac{2}{3}$. In short, we must take *Multiplication* and *Division* according to their established Definitions; about which I have said all that is necessary to make them be clearly understood.

C H A P. VII.

Of the more special Application of Fractions.

I Have in the preceding Chapter not only explained the general Principles and Operations in Fractions, but also made Application to particular things by Examples in all the Operations. These Examples are indeed Simple, and but pure Suppositions: And if we consider that Fractions require more Operation than absolute Numbers, it is unreasonable to bring them into Business without necessity. It remains then that I make a few further Reflections, and shew you how they necessarily occur in Business. In the first place observe, that in what we call mix'd Whole Numbers, the inferiour Species are indeed Fractions, but such as we may call *tacite ones*; because their Denominations are never express'd, it being thought more convenient to distinguish them by common Names than by numeral Denominations; yet these are always understood, and really applied in all Operations: And the Rules given about them in the *first Book*, are either the very same, or deduced from the general Rules of Fractions explained in this *Book*; and so have the same effect: For since each of these Rules is demonstrated to be true and

right, the Effect of each must be the same, when applied to the same things. But I shall also shew it by a short Comparison, thus :

In *Addition* of mix'd Whole Numbers, we add the Numbers of each Species by themselves without any further preparation, because they are Fractions of the same Integer, viz. the next higher Species, having all one common Denominator, (or numeral Relation to the next Species,) which is exactly the Rule of Addition of Fractions; and then if the Sum is greater than that Den^r, it's an Improper Fraction; and accordingly we do in effect reduce it to an Integral Number of that Species, in the very Addition, (which is equivalent to dividing it by the common Den^r) leaving the Remainder in the lowest Species to which it belongs. So Pence are 12th Parts of a Shilling, and therefore we carry the Number of 12's contained in the Sum of Pence to the Shillings; and so on.

In *Subtraction* we do the same in effect as the Rule for Fractions requires; for the like Species are Fractions having the same Den^r, (with respect to the next above,) and when the Number in the Subtrahend is least, we borrow Unity from the next Species, i. e. the Den^r, (as in *Examp. 7, 8, 9. Subtraction of Fractions.*)

In *Multiplication*, which is but a reiterated Addition, the Comparison is the same as in Addition, if the Work is performed by beginning at the lower Species, and multiplying upwards; and if we reduce first the Mix'd Number to a Simple Number of the lowest Species, then it is a Fraction of the highest, whose Den^r is the Relation betwixt the lowest Species and highest. And this Fraction is multiplied by multiplying its Num^r, (i. e. the Number produced by the Reduction,) and the Product makes an Improper Fraction; the Reduction of it being nothing else but finding its equivalent Mix'd Number, and again reducing the Integral Part to the higher Species.

In *Division*, if the Dividend is a Mix'd Whole Number, and the Divisor a simple Abstract Number, the Comparison is the same as in *Multiplication*, if we take the Method of reducing the mix'd Number to the lowest Species; and if we divide from the highest Species gradually, then it is the same as if we express'd each Species as a Fraction of the next above it, and divided each Member by itself. Again, if the Divisor and Dividend are both Applicate, whether both are mix'd, or only one, the Reduction of both to the same lowest Species, is making them both Fractions of the highest Species and of one Den^r, (viz. the Relation betwixt the highest and lowest,) and then dividing their Num^rs, which is according to the Rule of Fractions.

But again, as in Whole Mix'd Numbers each Species has a Relation to all the rest, so the several Members of a Mix'd Whole Number may be expressed as Fractions of the highest Species; and then if these are all added together, they will make with the Number of the highest Species, a Mix'd Fractional Number. For Example, 8 *l.* 6 *s.* 4 *d.* is equal to $1.8 + \frac{6}{20} + \frac{4}{240}$ and this reduced, is equal to $1.8\frac{76}{240}$. And when several Mix'd Numbers of one kind are thus express'd, the Addition or Subtraction by the Rules of Fractions, will bring out for the Sum or Remainder, a Number equal to what will arise from expressing the Sum or Remainder got by the common Rules, in the same fractional manner with respect to the highest. And the same thing will hold for the Products and Quotes in *Multiplication* and *Division*; for if this were not true, either the Rules given for fractional Operations, or those for Whole Mix'd Numbers must be false. But each of these are demonstrated to be true.

But again, *Observe*, that the common Rules for Mix'd Whole Numbers do make easier and distincter Work than what would happen by that way of expressing the inferiour Species, except upon certain Suppositions of their mutual Relations, as we shall immediately explain. But keeping to the common Subdivisions at present instituted, it is better to express them as Mix'd Whole Numbers, and use the Rules given about these in the first *Book*, and never bring in Fractions when they can be avoided: But this cannot always be done; for since Fractions necessarily arise from Imperfect Division, therefore they will

will unavoidably happen upon the Division of Numbers of any lowest Species, or of things for which no inferior Species are instituted.

Now, tho' the Operations with Mix'd Whole Numbers, according to the present Subdivisions, are easier than what would be if the Numbers of their inferior Species were express'd fractionally, and the general Method of Fractions applied; yet there is a Supposition in the Subdivision of Quantities, *i. e.* a certain Species of Fractions, according to which the *Division* being made, it would be more easy and convenient to express the lesser always as Fractions of the greater: and that Species is the *Decimal Fraction* already mentioned; whose Principles and Operations I shall first explain, and then more particularly their Use and Application.

C H A P. VIII.

Of DECIMAL FRACTIONS.

§. 1. DEFINITION.

IF we suppose any Integer divided into 10 Parts, and each of these again into 10 Parts, making of the Whole 100 Parts; and each of the last Parts again into 10 Parts, making of the Whole 1000, and so on: these Parts are called *Decimal Parts*; and any Number of them is called a *Decimal Fraction*: Whose Definition is therefore this, *viz.* a Fraction whose Den^r is 10, or 10 × 10, or 10 × 10 × 10, &c. *i. e.* 10, or 100, or 1000, &c.

Examp. $\frac{1}{10}$, $\frac{14}{100}$, $\frac{46}{1000}$, &c.

It is plain therefore that the Den^r of any *Decimal Fraction* is 1, with one or more o's on the Right-hand of it. And in this lies the essential Difference betwixt *Decimal Fractions* and all others. But there is also another Difference, which is in the *Notation* of them: For tho' they may be written in the *Vulgar Form*, yet from this Property of the Den^r, we have a Method of Notation different from and easier than the General or Common Way used in the preceding Chapters. And hence also we have Operations as Simple and Easy as those of Whole Numbers.

§. 2. NOTATION of DECIMALS.

THE *Numerator* and *Denominator* of a *Decimal Fraction*, whether Proper or Improper, being known, write first down the Num^r, then consider how many Cyphers, or o's belong to the Den^r; and beginning at the Right-hand or Place of Units of the Num^r, reckon towards the Left-hand one Figure or Place for every o in the Den^r. And if there are not as many, supply the Defect with o's, set on the Left-hand; and set before them a Point, (representing the 1 belonging to the Den^r) called the *Decimal Point*; which is therefore the Mark of a *Decimal Fraction*: As in the following Examples, written both in the *Vulgar* and *Decimal Form*.

Examp. 1. $\frac{3}{10}$ is .3 *Examp. 2.* $\frac{24}{100}$ is .24 *Examp. 3.* $\frac{426}{1000}$ is .0426

Examp. 4. $\frac{28}{10000}$ is .0028 *Examp. 5.* $\frac{46}{10}$ is 4.6 *Examp. 6.* $\frac{3467}{100}$ is 34.67

This *Notation* is arbitrary, and requires no Demonstration, but only to shew that the Num.^r and Den.^r are thus distinctly marked, tho' not altogether separated from one another: And this is very obvious; for the Num.^r is completely expressed, (what Cyphers are in some Cases, set on the Left-hand of it, changing nothing of its Value,) and because the Den.^r consists of a Number of o's on the Right-hand of 1, we want only to know how many are of these o's: and this we know by numbering the Places that stand on the Right-hand of the Point. Therefore

To read a Decimal which is written in its proper Form;

Take the whole Rank of Figures, which together make one Number, (*i. e.* the whole Rank excluding the o's that stand on the left of all) for the Num.^r; and for the Den.^r, reckon as many o's as there are Figures before the Point on the Right-hand. So .057 is $\frac{57}{1000}$, and .004607 is $\frac{4607}{1000000}$.

SCHOL. The same *Problems* and *Rules* of *Reduction* belong to Decimals as to any other Fractions, which need not be repeated. There are only these few particular things to be remarked, which are consequences of the Nature and Notation of Decimals, and the general *Rules* of *Reduction of Fractions* already explained; and which will serve as Principles for the Demonstration of the following *Rules* of Operation.

C O R O L L A R I E S.

1. A proper Fraction in Decimals can have no Figures standing on the Left-hand of the Point, but all upon the Right; for the Den.^r must necessarily have more Places than the Num.^r, and so the Point must fall without them on the Left-hand: And an Improper one must have Figures on both hands of the Point; for because the Den.^r cannot have more Places than the Num.^r, therefore the Point cannot fall without. *Examp.* .046, is Proper; and 4.62, is Improper.

2. A mix'd Decimal Number and its equivalent Improper Fraction have the same *Notation* in Decimal Form, and therefore require no Operation to reduce them; and so the Distinction can only be made in the reading them. For Example, $\frac{3467}{100} = 34 + \frac{67}{100}$, have the same Decimal Notation, *viz.* 34.67. For this is the Reduction of the Improper Fraction, by the General Rule; and it is plain the mix'd Number can have no other Notation, except that a Mark of Addition may be put betwixt the Integers and Fraction thus, $34 + .67$. But this Mark is superfluous. Hence any Decimal Expression, where there are Figures standing on the Left-hand of the Point, may be read either as an Improper Fraction, or a Mix'd Number; thus, as an Improper Fraction, taking the whole Rank as it stands (without minding the Point) for the Num.^r, and for the Den.^r reckon as many Cyphers as stand on the right of the Point; or as a Mix'd Number, by taking all the Figures on the left of the Point as a Whole Number, and those on the right as a Decimal Fraction. So this other *Example*, 234.08, is either $\frac{23408}{100}$, or $234 + \frac{8}{100}$.

3. It is manifest, that all Expressions wherein there are no Figures, but o's after the Point, are pure Whole Numbers composed of the Figures standing before the Point. So 34.00 is = 34. Yet such Expressions are equal to, and may be read as an improper Decimal Fraction, whereof the Num.^r is all the Rank of Figures on each side the Point, and the Den.^r has as many o's as stand after the Point; so the preceding is $\frac{3400}{100}$, which,

according to the Decimal Notation is 34.00. Wherefore a Whole Number is expressed in Decimal Form by setting any Number of 0's after it, and a Point between; which is to be read as a Fraction in the manner explained.

4. Cyphers standing in Places next the Right-hand of a Decimal, make the Fraction the same as if they were not there; so $.3400 = .34$. Because these Cyphers being reckoned both to the Numr and Denr, if they are taken away, do equally divide both; so $.3400$ being $\frac{3400}{10000}$, divide both Terms by 100, (or take two Cyphers from each) it is $\frac{34}{100}$, or $.34$.

5. Two Decimal Fractions having the same Number of Figures on the right of the Point, have the same common Denr; so $.34$ and $.06$ have the same Denr, viz. 100. Hence two or more Decimals are reduced to one Denr by adding as many Cyphers on the right of those that have fewer Places, till they have all an equal Number of Places. So these, $.4$, $.25$, $.067$, are equal to, $.400$, $.250$, $.067$; having a common Denr 1000.

6. Every proper Decimal Fraction is equal to the Sum of so many lesser ones, whose Numrs are the several significant Figures of the given Numr, their Denrs having as many 0's as there are Places from the Point to that Figure. For *Examp. 1.* $.34 = .3 + .04$, for $.3 = .30$, which has the same Denr with $.04$; and therefore that Sum is $.34$. *Ex. 2.* $.04608 = .04 + .006 + .00008$

§. 3. ADDITION of DECIMALS.

R U L E.

WHether the Numbers given are pure or mix'd Decimals, or some of them Whole Numbers, write them down under one another in such order that the Decimal Points stand all in a Column, and the Figures all in distinct Columns, in order as they are removed from the Point either on the Right or Left-hand; then beginning at the Column on the Right-hand, add the Figures in every Column together, and carry forward 1 for every 10 in the Sum, as in Whole Numbers; placing a Point in the Sum under the Points of the given Numbers.

Examp. 1.

$.24$
 $.378$
 $.057$
 $.9356$
 $.6827$

 2.2933

Examp. 2.

$.004$
 $.9$
 $.4067$
 $.08$
 $.235$

 1.6257

Examp. 3.

36.24
 450.058
 378.62
 8923.9
 42.007

 9830.825

If some of the Numbers given are Whole Numbers without a Fraction, the Work is the same, setting these Whole Numbers on the left of the Column of Points; and if the Sum of the first Column in the decimal Part is a Number of 10's, the 0 need not be written down, but proceed; and do the same, if the Figure to be set down in the next Column happens also to be 0. But mind that after a significant Figure comes in the Sum, such 0's belonging to the following Columns must not be neglected.

Exam^p. 4.

$$\begin{array}{r}
 468. \\
 24.06 \\
 48.724 \\
 2370. \\
 8.4 \\
 \hline
 2919.184
 \end{array}$$

Exam^p. 5.

$$\begin{array}{r}
 456 \\
 .08 \\
 37.604 \\
 478.26 \\
 .8 \\
 \hline
 517.2
 \end{array}$$

Exam^p. 6.

$$\begin{array}{r}
 67.004 \\
 8.206 \\
 .018 \\
 .4 \\
 .28 \\
 \hline
 75.908
 \end{array}$$

DEMONSTR. If you conceive as many o's to be set on the Right-hand of the Decimal Point in each of the Numbers added, till they have all an equal Number of Figures after the Point, they are thereby reduced to Fractions (Proper or Improper) having a common Den^r, (*Corol.* 5.) and the Sum of the Num^rs is the Num^r of the Sum sought to be applied to that common Den^r; but these o's adding nothing to the Sum, need not be filled up, and therefore the Sum of the Num^rs is truly found by the Rule; and by setting a Point in the Sum under the Column of Points in the Numbers added, the common Den^r is right applied, therefore the Rule is good.

Or we may deduce the Reason of this Rule thus: All the Figures of the fractional Parts standing in one Column, are Num^rs of Fractions having the same common Den^r, (by *Corol.* 6.) and from the nature of a Decimal Den^r, each 10 in the Sum of any Column is equal to a Fraction whose Num^r is 1, the Den^r having one o less than the Den^r of the Column added; *i.e.* $.010 = .01$. Therefore it is plain, that adding and carrying of every 10 from the several Columns according to the Rule, gives the true Sum.

SCHOL. In *Applicate Numbers*, the Decimals thus added must all refer to one Integer, or be reduced to that state; and in doing this, reduce always the higher Species to the lower, by multiplying the Numerator, or the lower to a higher by dividing the Num^r, if it can be done; and either way the Fraction found will be a Decimal. But if you reduce any other way, the Fraction will not always be a Decimal. *Exam^p.* .6 Yards + .08 Quarters, are, by Reduction, .24 qr + .08 qr. Or, .6 y^d + .02 y^d. But if instead of .08 qr, you put .07 qr. then if you reduce this to a Fraction of a Yard, it will not be a Decimal; for it can only be made $\frac{7}{80}$ Yards. And therefore reducing the Higher Denomination to the Lower is the General Rule that will keep the Expression in Decimals. But such Examples seldom occur; as we shall see afterwards in explaining the Use of Decimals.

§. 4. SUBTRACTION of DECIMALS.

R U L E.

ORder the Subtrahend and Subtractor the same way as directed in *Addition*, then subtract as in Whole Numbers, every Figure of the Subtractor from what stands over it in the Subtrahend, (supposing o to stand where there is no Figure in that Place of the Subtrahend;) and set a Point in the Remainder, in the same Column with the Points of the given Numbers; and when o's fall next the Right-hand of the Remainder, they need not be set down.

$$\begin{array}{r}
 \text{Ex. 1. } .84 \\
 .32 \\
 \hline
 .52
 \end{array}$$

$$\begin{array}{r}
 \text{Ex. 2. } 68.28 \\
 24.057 \\
 \hline
 44.223
 \end{array}$$

$$\begin{array}{r}
 \text{Ex. 3. } 48. \\
 9.86 \\
 \hline
 38.14
 \end{array}$$

$$\begin{array}{r}
 \text{Ex. 4. } 478. \\
 .027 \\
 \hline
 477.963
 \end{array}$$

Ex.

$$\begin{array}{r} \text{Examp. 5.} \quad 82.0642 \\ 29.7562 \\ \hline 52.308 \end{array}$$

$$\begin{array}{r} \text{Ex. 6.} \quad 234.075 \\ 82.04 \\ \hline 150.135 \end{array}$$

$$\begin{array}{r} \text{Ex. 7.} \quad 28.24 \\ 16. \\ \hline 12.24 \end{array}$$

The *Demonstration* of this Rule stands upon the same Principles as that for Addition. The same *Scholium* is also applicable here, which was made after Addition.

§. 5. MULTIPLICATION of DECIMALS.

R U L E.

TAKE the Numbers proposed (*i. e.* the Rank of Figures in each) as Whole Numbers, and multiply them one by the other as such, neglecting the o's that stand next the Left-hand, as useless in the Multiplication; then take the Number of Places or Figures, whatever they are, that stand after the Point, both in the Multiplier and Multiplied, and numbering as many Places from the Right-hand of the Product, set a Point before them; and if there are not as many, supply the Defect with Cyphers, and you have the Product duly expressed; which, according to the Circumstances of the Factors, may be either a Pure Fraction, (*Examp. 1, 2, 3.*) or a Mix'd one, (*Examp. 4, 5, 6.*) or a Whole Number, (*Examp. 7.*)

Examp. 1.

$$\begin{array}{r} .46 \\ .28 \\ \hline 368 \\ 92 \\ \hline .1288 \end{array}$$

Ex. 2.

$$\begin{array}{r} 42.78 \\ .00094 \\ \hline 17112 \\ 38502 \\ \hline .0402132 \end{array}$$

Ex. 3.

$$\begin{array}{r} .000486 \\ .024 \\ \hline 1944 \\ 972 \\ \hline .000011664 \end{array}$$

Ex. 4.

$$\begin{array}{r} 6.08 \\ .47 \\ \hline 4256 \\ 2432 \\ \hline 2.8576 \end{array}$$

Ex. 5.

$$\begin{array}{r} 723.246 \\ 48.002 \\ \hline 2169738 \\ 5785968 \\ 2872084 \\ \hline 24717.977728 \end{array}$$

Ex. 6.

$$\begin{array}{r} 578.32 \\ 64 \\ \hline 231328 \\ 346992 \\ \hline 37012.48 \end{array}$$

Ex. 7.

$$\begin{array}{r} 456 \\ 2.75 \\ \hline 2280 \\ 3192 \\ 912 \\ \hline 1254.00 \end{array}$$

DEMONSTR. The *Reason* of this Rule is obvious; for if you conceive the two given Numbers as Fractions, Proper or Improper, the Work of the Rule is plainly multiplying their Num^{rs} together, [which are the Rank of Figures proposed, taken as whole Numbers;] and applying to the Product a Decimal Den^r, equal to the Product of the given Den^{rs}: [Which is done by taking as many Places from the Product of the Num^r for Decimal Places, as there are in the Den^{rs} of both the Factors; for the Sum of these is the Number of Places in the Product.] And this Fraction is the Product sought, by the general Rule of multiplying Fractions, (*Chap. 5.*)

SCHOL. To multiply any Decimal (Pure or Mix'd) by 10, or 100, &c. (*i. e.* any Number expressed by 1, with o's, after it,) it is plain we have no more to do, but remove the Point as many Places towards the Right-hand, as there are o's in the Multiplier; and then, if all happen to be o's on the Left, they are useless and to be neglected.

Examp.

Examp. 1. $.46 \times 10 = 4.6$ *Ex.* 2. $.004 \times 100 = .4$ *Ex.* 3. $82.0375 \times 100 = 8203.75$

It is plain that this has the same Effect as the preceding Rule; according to which we should set as many o's after the Multiplicand as there are in the Multiplier; and then taking from the Product as many Decimal Places as are in the Multiplicand, (for we now suppose none in the Multiplier) the Point will be removed as many Places to the Right, as the Number of o's annex'd; and these being superfluous on the Right of a Decimal, need not be set down. But if there are not as many Places after the Point, the Defect must be supply'd with o's, and the Product will be a Whole Number. So $4.6 \times 100 = 460$. Or the Reason of this Practice is also clear by considering, that by setting the Point a Place nearer the Right, every Figure is thereby 10 times the Value it was; and consequently, so is the whole.

How to CONTRACT Multiplication.

When, betwixt the Multiplicand and Multiplier, there are more decimal Places than we incline to have in the Product, then we may find the Product true to as many decimal Places as we please, (or very nearly true) without producing all the rest of the Figures, which will in many Cases make a great Abridgement of the Work. For which take this

Rule. Consider how many decimal Places you would have in the Product; set the Figure in Unit's Place (*viz.* of Integers) of the Multiplier, under that decimal Figure of the Multiplicand, whose Den^r is what you would have in the Product, (*i. e.* under the 1st, 2^d, or 3^d, &c. Place after the Point, if you would have only one, two, or three decimal Places in the Product.) Then set the other Figures of the Multiplier in the reverse Order from that; and multiply by every Figure in the Multiplier: In doing which, begin only at the Figure of the Multiplicand, under which the multiplying Figure stands, neglecting all towards the Right. But at the same time consider what would have been carried from the Product of the preceding Figures on the Right, (which will be found in most Cases by multiplying the two next preceding Figures) that it may be added to the Product which is first written down. *Again,* Let all the partial Products be set under one another, so as the first Figures in each stand in one Column, and the rest in the same Order. *Lastly,* In adding these partial Products together, you must judge as near as you can what would have been carried from the preceding Columns, if we had neglected none of the Figures of the Multiplicand; so as to add that to the first Column written down. See the following *Examples*.

Examp. To multiply 47.32685 by 8.463, and have three Places of Decimals in the Product.

Operation at large.

47.32685
8.463
—
14198055
28396110
18930740
37861480
—
400.52712155

Abridged.

47.32685
3648
—
378614
18930
2839
141
—
400.527

As the Allowance for what may be carried from the Columns neglected is altogether a Guess, we may very often make the Product less than it ought to be, by 1 or 2 in the last Place; which can scarcely

be help'd otherwise than by making one or two more Columns than the Number of Decimal Places you would have in the Product, and then you may cut off the two last Places from the Product.

Examp.

Examp. 2. To multiply 463.25639 by 67.864, so as two Places of Decimals shall be true in the Product.

Operation at large.

$$\begin{array}{r}
 463.25639 \\
 67.864 \\
 \hline
 185302556 \\
 277953834 \\
 370605112 \\
 324279473 \\
 277953834 \\
 \hline
 31438.43165096
 \end{array}$$

Abridged.

$$\begin{array}{r}
 463.25639 \\
 468.76 \\
 \hline
 27795383 \\
 3242794 \\
 370605 \\
 27795 \\
 1853 \\
 \hline
 31438.43
 \end{array}$$

In this *Example* I have set the 7 which is in the Unit's Place of the Multiplier under the 3^d decimal Place of the Multiplicand, tho' I wanted only two Places in the Product, that by that other Place, the two first Places may be true.

Observe, If there be not as many decimal Places in the Multiplicand as are wanted in the Product, you must supply them with o's. As if in the preceding *Example* the Multiplicand were 463256.39 and I wanted four decimal Places in the Product, then I write the Multiplicand thus: 463256.3900 (or with one o more) and set the 7 of the Multiplier under the last o.

Again, If the Multiplier is all a decimal Fraction, imagine a o in the Unit's Place, and set the other Figures in order from that on the Left.

For the *Reason* of this Practice it is seen in the Comparison of the Work at large and abridged, which you see is but the former reversed.

§. 6. DIVISION of DECIMALS.

R U L E.

TAKE the Numbers proposed as Whole Numbers, *i. e.* such a Whole Number as the Rank of Figures would make, without regard to the Point; (in which View o's next the Left-hand will be altogether useless) and as such, divide the one by the other. If there is a Remainder, which is necessarily less than the Divisor, set a Cypher after it, and then divide again: But when the Remainder, with one o added, makes a Number less than the Divisor, set o in the Quote, and add another o; and so on, till the Remainder, with the o's added, make a Number greater than, (or equal to) the Divisor. And thus continue, adding Cyphers to the Remainder, and dividing, till there be no Remainder. But as this will not happen in every Case, the Division is to be thus carried on to a greater or lesser Number of Figures, according as the Circumstances of the Question require, as shall be further explained in the use of *Decimals*. Observe also, that if at the beginning of the Work, the Dividend makes a lesser Number than the Divisor, when both are considered as Whole Numbers, then set as many o's after it, till it be greater than (or equal to) the Divisor; and then begin the Division, proceeding with the Remainders, as before directed.

When the Division is finished, or carried on as far as you think fit, the Quote must be qualify'd in this manner; *viz.* Consider how many Decimal Places (or Figures after the Point on the Right-hand) there are in the Divisor, and also in the Dividend; (among which last are to be reckoned all the o's added to the Dividend, and to the Remainders: and if the Dividend is a Whole Number, the o's added are reckoned the Decimal Places of it.) Then, 1. If the Number is equal in both, the Quote is a Whole Number, (*Ex. 1.*) 2. If the Number in the Dividend is greatest, take the Difference, and separate as many for Decimal Places from the Right of the Quote, (supplying the Defect with o's) by a Point set before them; and then

then the Quote becomes either a *Proper Fraction* or *Improper*, i. e. a Mix'd Number. (*Examp. 2, 3, &c.*) 3. If the Number in the Divisor is greatest, take the Difference, and set as many 0's after the Quote, and take all for a Whole Number. (*Examp. 7, 8.*)

Examp. 1.

$$\begin{array}{r} .004 \overline{) .128} \quad (32 \\ \underline{12} \\ 008 \\ \underline{8} \\ 0 \end{array}$$

Ex. 2.

$$\begin{array}{r} .32 \overline{) .152} \quad (.475 \\ \underline{128} \\ 240 \\ \underline{224} \\ 160 \\ \underline{160} \\ 000 \end{array}$$

Ex. 3.

$$\begin{array}{r} .64 \overline{) .8476} \quad (1.3243 \\ \underline{64} \\ 207 \\ \underline{192} \\ 156 \\ \underline{128} \\ 280 \\ \underline{256} \\ 240 \\ \underline{192} \\ \text{Rem. } 48 \end{array}$$

Ex. 4.

$$\begin{array}{r} 2.7 \overline{) 109.35} \quad (40.5 \\ \underline{108} \\ 135 \\ \underline{135} \\ 000 \end{array}$$

Ex. 5.

$$\begin{array}{r} .8 \overline{) 2.04} \quad (2.55 \\ \underline{16} \\ 44 \\ \underline{40} \\ 40 \\ \underline{40} \\ 00 \end{array}$$

Ex. 6.

$$\begin{array}{r} 46 \overline{) .028} \quad (\\ \text{or} \\ 46 \overline{) .0280} \quad (6086 \\ \underline{276} \quad \text{True Quote,} \\ 400 \quad .006086 \\ \underline{368} \\ 320 \\ \underline{276} \\ \text{Rem. } 44 \end{array}$$

Ex. 7.

$$.024 \overline{) .48} \quad (20$$

Ex. 8.

$$\begin{array}{r} .567 \overline{) 2721.8} \quad (4800 \\ \underline{2268} \\ 4538 \\ \underline{4536} \\ \text{Rem. } 2 \end{array}$$

Of Valuing the Remainder, and compleating the Quote.

There remain yet, as a Part of this Rule, some further Considerations about the Remainder and Compleating of the Quote. For tho' in the Application and Use of Decimals, as we shall afterwards learn, the Remainder is neglected, yet what I am now to add, is not only fit to be known as a Part of the Theory, but necessary for our judging aright how far the Division ought to be carried on in different Cases, that the Defect of the Quote, arising from the Neglect of the Remainder, may not be too great. For this is certain, that where there is a Remainder the Division is not perfect; so that the Quote found, and qualify'd by the preceding Rule, will be deficient of the compleat Quote; and this Deficiency depending on the true Value of the Remainder, we shall first see how that is to be found, and then what is to be added to the Quote already found, to make the compleat Quote.

1. The Remainder is to be valued thus: Make it the Num^r of a Fraction, whose Den^r is that of the Dividend, (taking in all the Cyphers added in the Operation.) So in *Ex. 3.* the Remainder in its true Value is .000048; in *Ex. 6.* it is .0000044, and in *Ex. 8.* it is .2.

2. If you demand the compleat Quote, (when there is a Remainder) i. e. which multiplied by the Divisor will produce the Dividend (respecting their true Values) you may find it by the general Rule in *Chap. 6.* But if you would keep the Quote already found as one distinct Part, and would know what is to be added to it that the Sum may be the compleat Quote; then take the Remainder in its true Value (as above,) and divide it by the

the Divisor taken also in its true Value, by the Rule of *Chap. 6.* and that Quote is the thing sought. Or thus: First make a Fraction of the Remainder and Divisor (as Whole Numbers, without respecting their true Value) and then set as many o's after the Den^r as the Decimal Places of the Dividend exceed in number those of the Divisor; or if those in the Divisor be most, set as many Cyphers as the Difference after the Num^r: But if they are equal in Number, you have nothing more to do. And thus you have the thing sought, which will be the same as that found by the Rule of *Chap. 6.* only in lower Terms, as will easily appear by comparing them.

Thus in *Ex. 3.* the compleat Quote is $1.3243 + \frac{48}{240000}$; in *Ex. 6.* it is $.006086 + \frac{44}{48000000}$; in *Ex. 8.* it is $4800 + \frac{200}{387}$. See also the following *Examples*, where I have only written down the Quotes without the Operation.

Examp. 9. $.23) 46.8 (187 + \frac{5}{25}$.

Ex. 10. $.0432) 342.8 (7900 + \frac{2200}{432}$.

Ex. 11. $.008) 2.68 (330 + \frac{40}{8}$.

Ex. 12. $.08) 2742 (34200 + \frac{600}{8}$.

Observe again, that this additional Member to the Quote will always be a proper Fraction, when the Number of Decimal Places in the Dividend is equal to, or greater than that in the Divisor; for the Remainder which is the Num^r, is less than the Divisor which is the Den^r, and the o's are added to the Den^r: But if it's less, then it will be an improper Fraction in some Cases, (*Ex. 10, 11, 12.*) which shews,

that the Division being carried further on, the integral Number of the Quote would become greater; and particularly if you reduce that improper Fraction, then, as many Figures as its equivalent Whole Number contains, after so many more Steps in the Division, the Quote would have in it all the whole Number that can possibly belong to it, so that the additional Member will be after that a proper Fraction: So in *Ex. 10.* the compleat Quote being $7900 + \frac{2200}{432}$, and this Fraction being $= 35\frac{4}{3}$, makes the compleat Quote $7935\frac{4}{3}$. Again, if the additional Member is an improper Fraction, equal to some whole Number, it shews, that after so many more Steps as that Whole Number has Figures, the Division would have been perfect without a Remainder. So in *Ex. 11.* the additional Number of the Quote is $\frac{40}{8} = 5$, and the compleat Quote is 335; and in *Ex. 12.* the additional Member is $\frac{600}{8} = 75$, making the compleat Quote 34275.

Wherefore, that the additional Member of the Quote may be always a proper Fraction, (and so the first Part never want an Unit of the compleat Quote) carry on the Division till the Number of Decimal Places in the Dividend are equal to, or greater than that in the Divisor; unless the Division is finish'd without a Remainder before you come to that; for then the Quote found, and qualify'd according to the Rule, is compleat.

DEMONSTRATION of the preceding Rule.

The Divisor and Dividend being considered as Whole Numbers in the Operation, and the o's added to the Dividend and Remainder as belonging to the Dividend; then the Quote being found by the Rule of Whole Numbers, all we have to account for, is the qualifying of the Quote and Remainder, and the additional Member for completing the Quote. The Reason of which will easily appear, by comparing it with Multiplication.

We shall first suppose there is no Remainder, and then the Product of the Quote and Divisor is equal to the Dividend; but the decimal Places of any Product are equal to the Sum of the decimal Places in the Multiplier and Multiplicand: So the Number of decimal Places of the one Factor is the Difference of the Number in the Product, and in the other Factor, *i. e.* the Number of decimal Places in the Quote, must be equal to the Difference of the Numbers in the Dividend and Divisor, when the Number in the Dividend is greater, or equal to the Number in the Divisor, (which accounts for these two

Cases, see *Examp.* 1, 2, 3.) Again, When the Number of decimal Places in the Divisor is greater than in the Dividend, then the Quote found (without any Qualification) being multiplied into the Divisor, there would be more decimal Places in the Product than in the Dividend; wherefore that is not the true Quote: But now let as many integral Places of o's as the Difference of the Number of Places in the Divisor and Dividend, be annex'd to the Quote; and this multiplied into the Divisor, there will be the same Number of decimal Places in the Product as before. But as many of these Places next the Right, as the forefaid Difference of decimal Places in the Divisor and Dividend, being o's, because of the o's annex'd to the Quote, they don't increase the decimal Part; and therefore being cut away from the Product, they leave no more decimal Places in the Product than in the Dividend, the Product being the very same Rank of Figures: therefore the Quote is truly qualify'd. So if .48 is divided by 24, the Quote is 2; but if the given Numbers are .024 and .48, the Quote must be 20, for $.024 \times 2 = .048$, which makes one more decimal Place in the Product than in the Dividend; therefore that is not the true Quote; but reckoning this 20, the Product is $.480 = .48$

In the next place, suppose there is a Remainder; then, that the Quote and Remainder are duly qualify'd by the Rule, will easily appear thus: The Product of the Quote and Divisor is equal to the Dividend, after the Remainder is subtracted out of it, (taking them all as Whole Numbers;) therefore the Remainder added to that Product, makes the Dividend. But now the Quote being qualify'd, as in the Rule, (*i. e.* with regard to the number of decimal Places in the Dividend and Divisor) the Product must necessarily have as many decimal Places as the Dividend, otherways the Quote (which is qualify'd with a Regard to the decimal Places of the Dividend) would not be the true Quote out of that Product; which shews the Reason of the Rule for qualifying the Quote in this Case. Then for the Value of the Remainder, it's plain its Figures must be of the same Value with the Places of the Dividend of which it's the Remainder, which are the last Places on the Right-hand: Or also thus; That the Remainder added to the Product of the Divisor and Quote (in their true Values) may make up the Dividend, it's evident it must be of the same Value with the Places of the Dividend to which it's added; and these are the last Places on the Right-hand, which make the Num^r of a Fraction, whose Den^r is that of the Dividend; wherefore the Remainder must be so also, (which, according to different Circumstances, will be a Whole Number, or a Fraction, or Mix'd.)

Another DEMONSTRATION.

The *Demonstration* of all that relates to th's Rule, (*viz.* for both Members of the compleat Quote when there is a Remainder) may also be easily deduced from the General Rule in *Chap.* 6. and shewn to be the same. Thus, When the Division is finished, or you have put a Stop to it, consider the Dividend and Divisor as Fractions, Proper or Improper, as they happen (reckoning always the o's, added to the Dividend and Remainders, to belong both to the Num^r. and Den^r of the Dividend.) And let $\frac{a}{n}$ be the Divisor and $\frac{b}{m}$ the Dividend, (wherein the n and m are both decimal Den^{rs}, or the one of them such, and the other 1, as happens when there is no decimal Place in that Term, but all a Whole Number.) Then by the General Rule, the compleat Quote of $\frac{b}{m}$ divided by $\frac{a}{n}$, is $\frac{bn}{am}$ equal to $\frac{b}{a} \times \frac{n}{m}$. Now $\frac{b}{a}$ is the Quote of the two Num^{rs}, which if it's a Whole Number (there being no Remainder) call it q , and if there is a Remainder, let it be r ; then is $\frac{b}{a} = q + \frac{r}{a}$ (the very thing found by the Rule of Division of Decimals:) But this

this must be multiplied by $\frac{n}{m}$, *i. e.* both Parts, q , and $\frac{r}{a}$, if there is a Remainder; which is the same very thing in effect that this Rule directs to be done, for qualifying the Quote first found, and then compleating it. For it's plain, if $n = m$ (*i. e.* if the decimal Places in the Divisor and Dividend are equal in number) the Quote found is not thereby altered, and therefore $q + \frac{r}{a}$ is the compleat Quote.

Again, Suppose n and m unequal, then by cutting away an equal Number of o's from n and m , they are equally divided. Now therefore if m (the Den^r of the Dividend) has more o's than n (the Den^r of the Divisor) the Fraction $\frac{n}{m}$ is equal to a Fraction whose Num^r is 1, and its Den^r a decimal one, having as many o's as the Difference of the Number of o's in n and m . For *Examp.* $\frac{n}{m} = \frac{100}{1000} = \frac{1}{10}$. But to multiply by such a Fraction, is to divide by its Den^r; *that is*, plainly to divide $q + \frac{r}{a}$ (the Number found by the Operation) by a decimal Den^r, having as many o's as the decimal Places of the Dividend exceed in Number those of the Divisor. And this Division is done by setting off so many decimal Places in the Part q ; and in the Part $\frac{r}{a}$, by setting as many o's after the Den^r a , which is the 2^d Case in the Rule.

Again, let n be greater than m , then is $\frac{n}{m}$ equal to a Whole Number consisting of 1, and after it as many o's as those in n exceed those in m . *Examp.* $\frac{n}{m} = \frac{1000}{10} = 100$. But $q + \frac{r}{a}$, is to be multiplied by this; which is done by setting as many integral Places of o's both after q , and after r , as there are in $\frac{n}{m}$ so reduced; *i. e.* as the decimal Places of the Divisor exceed in Number those of the Dividend. Which is the last Case of the Rule.

We have here demonstrated the Reason for qualifying the Quote, and compleating it; but have not considered the true Value of the Remainder by itself, which is that in which it must be added to the Product of the Divisor and Quote, taken in their true Values to make up the Dividend, (according to its true Value.) But this is done in the former Demonstration.

Of the Use and Application of DECIMAL FRACTIONS.

WE have already observed, that the great Benefit proposed by Decimal Fractions, is a more simple and easy Operation than what Vulgar Fractions, taken either in their proper Form, or as mix'd Integers, do require. We shall consider how the Application is made for answering that End, and how far it's a real Advantage.

In the first place, this is very evident, that if instead of the Subdivision of Coins, Weights and Measures, (and other kind of Quantities useful in Society) which now obtain, there were one standard superior Species, and all the Subdivisions were Decimals, whether the several Parts were also distinguished by Names, or only by their decimal Denominations, it were the same thing to the purpose; then the Common Operations would be as simple and easy as Whole Numbers. The Rules and Reasons of which are, I hope, compleatly explained in the preceding Part. But supposing this were so, yet either we could not entirely avoid the Consideration of Vulgar Fractions, or we must admit of some

Inaccuracies in Calculations, which are unavoidable with Decimals; and which will be of more or less Consequence in different Circumstances. For we have seen that Decimals will have Remainders (because every Number is not an Aliquot Part of every other) and then the Quote is not compleat without bringing in a Vulgar Fraction; and therefore if we take the Quote without this Correction, it's less than just according to the Value of the Remainder, or rather the Value of the Vulgar Fraction that's necessary to compleat it. Now, if the Number found by this Division is the final Answer of a Question, which is to be applied in no further Calculation, then if it is brought so low as to be less than any Quantity of that kind that is used, (for Example, the smallest real Coin or Weight, &c. that has any Name or distinct Being in Society) then the Defect is not to be complained of; because if you do compleat the Quote, the additional Part is of no use: But if a Quote is to be further employ'd in Calculation, especially if it's to be multiplied, the Defect may become considerable; and it will be the more so, as the Multiplier is greater, and also according to the Value of the Integer. Now the only Remedy for this, while we use none but decimal Fractions, is to bring the Division very low; *i. e.* carry it on till the Den^r be very large, and consequently what is deficient be very little: and this is to be regulated according to the Circumstances above-mentioned; for which you'll find more particular Rules afterwards. But then this Inconveniency will frequently happen, That by this means we shall have very large Numbers to work with, which will prove more troublesome than the Method of Vulgar Fractions. These things we shall find more particularly exemplify'd afterwards.

Again, Tho' decimal Subdivisions are not in common use, yet they may be applied by a Reduction of the common Species to Decimals, and these back again to the other: I shall therefore explain this Reduction, and then by particular *Examples* shew the Application, with such Remarks as may give a general View of the Conveniences and Inconveniences of Decimals, and consequently help to judge where they are preferable or not to the common Method.

P R O B L E M I.

To reduce a Vulgar Fraction to a Decimal.

RULE. To the Num^r of the given Fraction add one or more o's, as Decimal Places, till it be greater than (or equal to) its Den^r; then divide by the Den^r, adding o's to the Remainders, and carrying on the Division (as directed in *Division of Decimals*,) till o remain, or as far as you please: then make the Quote a Num^r, and apply to it a Decimal Den^r, with as many o's as the Number of o's added to the Dividend and Remainders. This Decimal is exactly equal to the given Vulgar Fraction, if there is no Remainder in the Division; but if there is still a Remainder, that Decimal is deficient by a Compound Fraction, the one Member of which is a Simple Fraction whose Num^r is the Remainder, and its Den^r the Divisor; and the other Member is a Fraction whose Num^r is 1, and its Den^r is that of the Decimal already found. Or it is a Simple Fraction whose Num^r is the Remainder, and its Den^r the Divisor multiplied by the Den^r of the Decimal Fraction found; *i. e.* having as many o's prefix'd to it as belong to the Den^r; wherefore the more Places the Decimal found has, the less is the Defect.

The *Reason* of this Rule you have in the *Division of Decimals*; for the Dividend with the o's added is an Improper Decimal. Or you may take it from *Prob. 7. Chap. 2.* for the o's added to the Num^r and Remainder belong to the Decimal Denominator of the Fraction sought, by which the Num^r of the given Fraction is multiplied, and the Division made finds the correspondent Num^r according to that Rule, to which the Den^r is ap-

plied according to the Notation of Decimals; and what this Decimal Fraction is deficient, is also found according to the same Rule.

Ex. 1. $\frac{1}{2} = .5$ Ex. 2. $\frac{3}{4} = .75$ Ex. 3. $\frac{1}{16} = .0625$ Ex. 4. $\frac{7}{13} = .5384, \text{ \&c.}$

Operation.

$$2 \overline{) 10} (5$$

Oper.

$$\begin{array}{r} 4 \overline{) 30} (75 \\ \underline{28} \\ 20 \\ \underline{20} \end{array}$$

Oper.

$$\begin{array}{r} 16 \overline{) 100} (625 \\ \underline{96} \\ 40 \\ \underline{32} \\ 80 \\ \underline{80} \end{array}$$

Oper.

$$\begin{array}{r} 13 \overline{) 70} (5384 \\ \underline{65} \\ 50 \\ \underline{39} \\ 110 \\ \underline{104} \\ 60 \\ \underline{52} \\ \text{Rem. } 8 \end{array}$$

In the 4th *Examp.* there is a Remainder 8; and so the Quote wants $\frac{8}{130000}$ of the complete Value of the given Fraction; i. e. $\frac{7}{13} = .5384 + \frac{8}{130000}$; this being the true Value of the Remainder.

Observe, The Decimal may be found all at once in the Division, by first setting a Point in the Place of the Quote; then if one 0 does not make the Num^r equal to the Den^r, set 0 after the Point; and if another 0 makes it still less than the Den^r, set another 0 in the Quote, and so on: *That is,* set as many 0's after the Point in the Quote as the Number of 0's, which being set on the Right-hand of the Num^r, leaves it still less than the Den^r; and then after these 0's comes the Num^r of the Decimal, found by setting one 0 more on the Right-hand of the Num^r which gives the first significant Figure of the Quote, and by 0's gradually annex'd to the Remainders, the Work is carried on, and the Num^r and Den^r of the Decimal sought are thus both together formed; the Reason of which is manifest from the way of finding and applying the Den^r to the Num^r. See this *Example*.

Ex. $\frac{1}{128} = .0078125$

Operation.

$$\begin{array}{r} 128 \overline{) 1000} (.0078125 \\ \underline{896} \\ 1040 \\ \underline{1024} \\ 160 \\ \underline{128} \\ 320 \\ \underline{256} \\ 640 \\ \underline{640} \end{array}$$

COROLLARY 1. In dividing any Whole Number by another, when there is a Remainder, instead of making a Vulgar Fraction of it, we may turn it into a Decimal equal or nearly equal to it, by carrying on the Division with 0's added to the Remainders, (in the manner taught in *Division of Decimals*, or in the preceding *Problem*,) till 0 remain; then the Decimal is equal to the Vulgar Fraction: or till there be many Decimal Places, and then it is nearly equal to it. But how far it ought to be carried on, depends upon Circumstances of the Application (See the following *Scholium* 2.) And after the Integral Quote found, set a Point; so that all the Figures that come after, are Decimal Places: As in the following *Example*.

$$\begin{array}{r}
 28 \overline{) 7645} \quad (273.0357 \\
 \underline{56} \\
 204 \\
 \underline{196} \\
 85 \\
 \underline{84} \\
 100 \\
 \underline{84} \\
 160 \\
 \underline{140} \\
 200 \\
 \underline{196} \\
 \text{Rem. } 4 \text{ whose} \\
 \text{true Value is } .0004
 \end{array}$$

COROL. 2. When of a simple applicate Whole Number it is proposed to find a certain Part, instead of reducing the Remainders in the Division to lower known Species, we may carry on the Division decimally; and so all the Numbers of inferiour Species that would arise by reducing and dividing are thus turned into a decimal Fraction, (the Design and Use of which you will hear of more particularly afterwards.) If the Division does not soon come to an end, carry it on as Circumstances make it necessary. See the Rule in *Schol. 2.* following.

SCHOLIUM I.

That every Vulgar Fraction is not reducible to a determinate Decimal, (*i. e.* where there is no Remainder in the Division,) we know in fact by Examples; in which we find this certain Mark That the Division will never come to an end, *viz.* that there happens a Remainder which is the same with a former Remainder; in which Case it is not only certain that the Division will never have an end, but this we know also that the remaining Figures of the Quote must necessarily be a continual Repetition of the same Figures (in the same order) that stand already in the Quote from that one which proceeded from that former Remainder; as in these Examples.

Examp. 1.

$$\begin{array}{r}
 3 \overline{) 20} \quad (66, \&c. \\
 \underline{18} \\
 20 \\
 \underline{18} \\
 2, \&c.
 \end{array}$$

To reduce $\frac{2}{3}$ to a Decimal, it is .666, &c. (the 6 being always repeated,) for it is manifest the same Figure 6 will always arise in the Quote, because it is the same Dividend 20.

Examp. 2. To reduce $\frac{22}{53}$, it is .42626, &c. the 26 being still repeated; for the Remainder 21 which the Operation is stop'd being the same as a former which was the very first Remainder, it is plain that carrying on the Work, we should have for the next two Figures in the Quote 26, and so on still 26 *in infinitum*. Therefore whenever this happens in any Case, we need proceed no further, but observing what Figures in the Quote would be repeated, take as many of them, or the whole of them, as many times as we think fit.

Operation of Ex. 2.

$$\begin{array}{r}
 495 \overline{) 2110} \quad (.426, \&c. \\
 \underline{1980} \\
 1300 \\
 \underline{990} \\
 3100 \\
 \underline{2970} \\
 \text{Rem. } 130
 \end{array}$$

Such Decimals are very properly called Circulating Decimals, because of the continual Return of the same Figures; and may be called Indeterminate or Infinite Decimals, because they can never come to an end: as we also call those which are the Effect of a Reduction which has no Remainder, Finite or Determinate Decimals. *Observe* also, that these Infinite Decimals may be reckon'd as complete, because tho' they are composed of an infinite Series of Fractions, yet there is a certain and known Order in the Progression of the Series, from the constant Repetition of the same Figures, whereby

whereby they are capable of being managed in Operations so as nothing shall be wanting. But the Demonstration of the Theory and Rules of Operation with such Fractions requires other Principles than have yet been explained, and must therefore be referred to another place: (See *Book 5. Chap. 4.*) wherein you'll find it demonstrated, that every Vulgar Fraction will reduce to a Decimal, either finite or circulating.

SCHOLIUM II.

Tho' some Vulgar Fractions will become finite Decimals, yet if these have a great many Places, the Use of them will become very inconvenient and tedious in Practice. Also, tho' we have Rules for managing Circulating Decimals without any Defect, yet the same Inconveniency will arise in these, when they circulate upon many Figures, or when the Circulation begins at a great distance from the Point; therefore it is sufficient for common Use to carry the Reduction so far only, as that the Defect be inconsiderable; (for the further the Reduction is carried, the Defect is the less :) in order to which, I shall here shew you how

PROBLEM II.

To carry the Reduction of a Vulgar Fraction so far, that the Decimal found shall want less than any assigned Fraction.

RULE. Let the assigned Fraction be represented by $\frac{a}{b}$, if a Decimal is carried to so many Places after the Point, as are expressed by b ; the defect of that Decimal cannot be equal to $\frac{a}{b}$: For it cannot want a Fraction whose Num^r is 1, and its Den^r that of the decimal Quote already found, which we may express $\frac{1}{100\text{ } \&c.}$ the Den^r having as many o's as b has Figures. Since in that Case the Figure in the last Place found would necessarily be greater than it is by 1; or, because by the preceding Rule of this Problem the defect is only a compound Fraction, whereof one Member is this Fraction $\frac{1}{100\text{ } \&c.}$ which Defect is therefore less than $\frac{1}{100\text{ } \&c.}$ and this is evidently less than $\frac{a}{b}$; since 100, &c. having as many o's as b has Figures, must be a greater Number; and a is not less than 1: Wherefore $\frac{1}{100\text{ } \&c.}$ must be less than $\frac{a}{b}$.

To apply this more particularly: Consider to what Integer any Decimal refers; reduce that Integer to the lowest known Denomination; if the Decimal has as many Places after the Point, as that Number of the lowest Denomination which is equal to the Unit to which the Decimal refers, then the Decimal does not want the Value of an Unit of that lowest Denomination. And if the proposed Decimal were again to be multiplied by any Number, then to make it so that the Product shall not be deficient by an Unit of the lowest Denomination, make it have as many Places after the Point, as the Sum of the Number of Figures in the proposed Multiplier, and in the Number of Units of the lowest Denomination which makes an Unit of the Denomination to which the Decimal refers.

For *Examp.* If a Decimal of 1 l. has 3 Places after the Point, it does not want $\frac{1}{1000}$ of it, therefore does not want 1 Farthing, which is $\frac{1}{960}$ of it. And if the Decimal is again to be multiplied by a Number of 5 Places, let the Decimal have 8 (= 5 + 3) Places, and the Product shall not want 1 f. For being carried to 8 Places, it cannot want $\frac{1}{10000000}$ of 1 l. which is $\frac{1}{10000000}$ of $\frac{1}{10000}$. But $\frac{1}{10000}$ is less than 1 f. and hence

hence $\frac{1}{1000000}$ of $\frac{1}{100000} l.$ is less than $\frac{1}{1000000}$ of 1 *l.* Consequently if the Decimal carried to 8 Places is multiplied by a Number of 5 Places, (which is less than 100000) the Product cannot want 1 *l.* The Universality of this Reason for all Cases is manifest.

P R O B L E M III.

To reduce Integral Numbers of inferior Denominations to the Decimal of a higher.

CASE I. To reduce a Simple Number.

R U L E. Express it first as a Vulgar Fraction of the higher, by making itself the Num^r, and taking for the Den^r the Number of the inferior Denomination that is equal to 1 of the higher. Then reduce this Vulgar to a Decimal Fraction, by the last Problem.

Examp. 1. To express 5 *lb.* in the Decimal of a Pound: First, it is $\frac{5}{20} l.$ and this again is reduced to .25 *l.*

Operation.

$$\begin{array}{r} 112 \overline{) 300} \quad (.0267857 \\ \underline{224} \\ 760 \\ \underline{672} \\ 880 \\ \underline{784} \\ 960 \\ \underline{896} \\ 640 \\ \underline{560} \\ 800 \\ \underline{784} \\ \text{Rem. } 16 \end{array}$$

Examp. 2. 3 *lb.* *Averdupoise* Weight the greater, to the Decimal of a hundred Weight. It is $\frac{3}{112} C.$ and this is again .02678 *C.* Here the *Division* is imperfect, and we are to carry it on less or more as Circumstances require, according to the preceding Directions.

Another Method.

$$\begin{array}{r} \text{lb.} \\ 28 \overline{) 3.00} \\ \underline{4} 3.00 \\ 7 .7500000 \\ \underline{4} .1071428 \\ .0267857 \end{array}$$

It will be in most Cases easier to divide gradually from one Species to another, as in the Margin; where 3 *lb.* is divided by 28, (at two Steps, viz. by 4 and 7.) to bring it to the Decimal of 1 *qr.* and this again by 4, which brings it to the Decimal of 1 *C.*

CASE II. To reduce a Mix'd Number.

R U L E. Reduce each of the Numbers by the first Case, and then add their Decimals together: Or reduce the mix'd Number to a simple Number of the lowest Species, and then turn that into a Decimal.

Examp. To reduce 9 *lb.* 6 *d.* to a Decimal of 1 *l.* it is .475 *l.* by adding .45, (the Decimal equal to 9 *lb.*) and .025, (that equal to 6 *d.*) or by reducing 9 *lb.* 6 *d.* viz. 114 *d.* equal to $\frac{214}{450} l. = .475 l.$

Another Method to find the Decimal of a mix'd Number. Reduce the Number of the lowest Species to the Decimal of the next above; (whether there be any Number of that Species in the Question, or not,) add to it the Number of that Species in the Question, (if

(if there is any) and reduce the Sum to the next higher Species; adding to the Number found, the Number of that Species given in the Question; and go on so till you come to the proposed Integer.

Examp. To reduce 4 *lb.* 7 *d.* 3 *f.* to the Decimal of a Pound: Makes .2322916 *l.*
 Thus, 3 *f.* is .75 *d.* to which add 7 *d.* Then is 7.75 equal to
 4 | 3.00
 12 | 7.750000
 20 | 4.6458333, &c.
 .2322916, &c.

SCHOLIUM.

Concerning the Construction and Use of DECIMAL TABLES.

In order to the Application of Decimals, we ought to have ready calculated the Decimal of any Integer of *Money, Weight, Measure, &c.* answering to every Number, Simple or Mix'd, of inferiour Denomination, and of less Value than that Integer; which Decimals being orderly collected and disposed, make what we call *Decimal Tables*, by which any Decimal required may be readily found, or also the Value of any given Decimal in known inferiour Species.

As to the Construction of these Tables, it would certainly be a very tedious Work to find every Decimal by a separate Application of this *Problem*, [tho' this is a complete general Rule.] There are other Methods to shorten and make that Construction easier; which I shall here explain as far as the Principles already taught do permit.

RULE. Find the Decimal equal to an Unit of the lowest Species; and if that is a determinate Decimal, then from it, as the Root, the Decimals of all the rest may be found accurately by *Addition*. Thus, double the Root, that gives the decimal Fraction for 2 of that lowest Species; then add the Decimal of 1 and 2, the Sum is the Decimal of 3, and so on by adding still the last Decimal to the first, till you come to a Number of that Species equal to an Unit of the next Species above; then make that the First or Root of all the Decimals of that Species, making them up the same way as the last, *i. e.* doubling the first for the Decimal of 2 (of that Species,) add the Decimal of 1 and 2 for 3, and so on in this manner go through all the Species till you come to the Integer itself; and if you add the Decimal of the Number next less than the Integer to the Root, the Sum will be Unity in the Place of Integers.

If the Root, carried to a certain Number of Places, is not determinate, then is it a deficient Decimal; and if we make up the Table from that Root, all the other Decimals in the Table are also deficient, wanting gradually more and more from the Root upwards, so that the Number that comes against the Integer will be a Decimal. But the more Places the Root is carried to, the less is the defect in that and every other Part of the Table. And that there may not be wanting in any of these Decimals an Unit of the lowest Species, or any Part you please of such an Unit, follow the Directions already given.

Observe again, That as all Vulgar Fractions become Decimals, which are either determinate or circulate, so there is an easy way of making up the Tables by Addition from a Root which has one or more circulating Figures, (and that by using only the first Period of them) so as all these Decimals which would be found determinate by a separate Reduction, shall come out so in the Table, and all the rest circulate upon the same Number of Figures in the same Places as the Root does. But as the Reason of this Method

depends upon the particular Doctrine of circulating Decimals, and both the Method and Reason will be easily understood when you learn that Doctrine in *Book 5. Chap. 4.* I shall say no more of it here. And only *observe* these few things.

1. That to bring some Vulgar Fractions to a Decimal, determinate or circulating, will be such a long Work, that it's more convenient to take it imperfect with a less number of Places.

2. As to the following Tables *observe*, That the Roots of some of them are determinate; whence all the other Decimals of such Tables are also determinate; and are known by their wanting this Sign $+$ after them. In others the Root circulates, and so the Table is made up not by the Method of Addition simply, but with a due regard to that Circulation; so that the Decimals in the Table are these which would be found by calculating each separately, and carrying it to the same Number of Places.

3. But again *observe*, That the Roots of some of the Tables circulating upon a single gure, that Decimal is taken as far as the first Figure of the Circulation; and what Decimals in such Tables do circulate, are marked by the Sign $+$ after the circulating Figure; those which want it being determinate.

4. In the last place *observe*, That for the Tables of *Averdupoise Weight the greater*, and of *Time*, the Root was carried on to a Circulation, and the Table made up with a regard to that: But these Decimals running to many Places, you have here only the first seven Places, which are enow for common Use. And for *Money*, I have made two different Tables; the Manner and Design of which I have explained with the Table.

DECIMAL TABLES of MONEY, WEIGHTS, and MEASURES.

TABLE I. MONEY.

The Integer 1 Pound.

Farth.	1	.0010416	+
	2	.002083	+
	3	.003125	
Penny	1	.00416	+
	2	.0083	+
	3	.0125	
	4	.016	+
	5	.02083	+
	6	.025	
	7	.02916	+
	8	.03	+
	9	.0375	
	10	.0416	+
	11	.04583	+
Shilling	1	.05	
	2	.1	
	3	.15	
	4	.2	
	5	.25	
	6	.3	
	7	.35	
	8	.4	
	9	.45	
	10	.5	
	11	.55	
	12	.6	
	13	.65	
	14	.7	
	15	.75	
	16	.8	
	17	.85	
	18	.9	
	19	.95	

Another TABLE of MONEY.

Farth.	1	.0010416666
	2	.0020833332
	3	.0031249998
Penny	1	.0041666664
	2	.0083333328
	3	.0124999992
	4	.0166666656
	5	.0208333320
	6	.0249999984
	7	.0291666648
	8	.0333333312
	9	.0374999976
	10	.0416666640
	11	.0458333304
Shilling	1	.0499999968
	2	.0999999936
	3	.1499999904
	4	.1999999872
	5	.2499999840
	6	.2999999808
	7	.3499999776
	8	.3999999744
	9	.4499999712
	10	.4999999680
	11	.5499999648
	12	.5999999616
	13	.6499999584
	14	.6999999552
	15	.7499999520
	16	.7999999488
	17	.8499999456
	18	.8999999424
	19	.9499999392
	20	.9999999360

Observe, This second Table of Money I have made for no other end but to shew, that if the Root of any Table is not determinate, yet being taken to many Places, the Error will be very little; for the Table being here carried to 20 *lb*. this instead of being equal to 1 *l*. [as it would be had the Root been determinate, or the Numbers calculated with a due regard to that, as in the first Table,] it is a Decimal of 1 *l*. but which wants less than $\frac{1}{1000000}$ Part; the rest wanting less as they stand nearer to 1 *l*. For the Error grows from 1 *l*. to 1 *l*.

TABLE II. Troy Weight,

The Integer	I ounce.	
Grains	1	.002083 +
	2	.00416 +
	3	.00625 +
	4	.0083 +
	5	.010416 +
	6	.0125 +
	7	.014583 +
	8	.016 +
	9	.01875 +
	10	.02083 +
	11	.022916 +
	12	.025 +
	13	.027083 +
	14	.02916 +
	15	.03125 +
	16	.03 +
	17	.035416 +
	18	.0375 +
	19	.039583 +
	20	.0416 +
	21	.04375 +
	22	.04583 +
	23	.047916 +

Penny }
Weight } 1 | .05

The Table for Penny Weights is the same as that for Shillings with respect to 1 Pound.

TABLE V. Of Liquid Measure.

The Integer	I gall.
q ^r of a Pint	1 .03125
	2 .0625
	3 .09375
Pint	1 .125
	2 .25
	3 .375
	4 .5
	5 .625
	6 .75
	7 .875

TABLE III. Averdupoise Weight the Lesser.

The Integer	I lb.
q ^r of a Dram	1 .00097656 +
	2 .00195313 +
	3 .0029296 +
Dram	1 .00390625
	2 .0078125
	3 .01171875
	4 .015625
	5 .01953125
	6 .0234375
	7 .02734375
	8 .03125
	9 .03515625
	10 .0390625
	11 .04296875
	12 .046875
	13 .05078125
	14 .0546875
	15 .05859375
Ounce	1 .0625
	2 .125
	3 .1875
	4 .25
	5 .3125
	6 .375
	7 .4375
	8 .5
	9 .5625
	10 .625
	11 .6875
	12 .75
	13 .8125
	14 .875
	15 .9375

TABLE IV. Averdupoise Weight the Greater.

The Integer	I Cwt.
q ^r of an Ounce	1 .0001395
	2 .0002790
	3 .0004185
Ounce	1 .0005580
	2 .0011160
	3 .0016741
	4 .0022321
	5 .0027901
	6 .0033482
	7 .0039062
	8 .0044642
	9 .0050223
	10 .0055803
	11 .0061383
	12 .0066964
	13 .0072544
	14 .0078125
	15 .0083705
Pound	1 .0089285
	2 .0178571
	3 .0267857
	4 .0357142
	5 .0446428
	6 .0535714
	7 .0625, Exact.
	8 .0714285
	9 .0803571
	10 .0892857
	11 .0982142
	12 .1071428
	13 .1160714
	14 .125, Exact.
	15 .1339285
	16 .1428571
	17 .1517857
	18 .1607142
	19 .1696428
	20 .1785714
	21 .1875, Exact.
	22 .1964285
	23 .2053571
	24 .2142857
	25 .2232142
	26 .2321428
	27 .2410714

TABLE VI. Of Dry Measure.

The Integer	1 qr Chald.
Pint 1	.001953
2	.003906
3	.005859
qr of a Peck 1	.0078125
2	.015625
3	.0234375

For Pecks and Bushels, they have the same Decimals as qr of Pints, and Pints in Liquid Measure with respect to 1 Gallon.

TABLE VII. Of Long Measure.

The Integer	1 yd.
qr of Nails 1	.015625
2	.03125
3	.046875
Nails 1	.0625
2	.125
3	.1875
qr of Yard 1	.25
2	.5
3	.75

TABLE VIII. Of Long Measure.

The Integer	1 foot.
qr of an Inch 1	.02083
2	.0416
3	.0625
Inches 1	.083
2	.16
3	.25
4	.3
5	.416
6	.5
7	.583
8	.6
9	.75
10	.83
11	.916

TABLE IX. Of Time.

The Integer	1 Year.
Days 1	.0027397
2	.0054794
3	.0082191
4	.0109589
5	.0136986
6	.0164383
Weeks 1	.0191780
2	.0383561
3	.0575342
4	.0767123
5	.0958904
6	.1150684
7	.1342465
8	.1534246
9	.1726027
10	.1917808
11	.2109589
12	.2301369
Quarter 1	.25
2	.5
3	.75

In this Table of Time, the Decimal for 1 Day is taken by the 365th Part of a Year; and so the Table is carried on to 12 Weeks, or 84 Days. Then as I reckon 13 Weeks to 1 Quarter of a Year, so after 12 Weeks comes next 1 Quarter. But that Decimal and the following are taken accurately, which would not happen if they were continued from the preceding. So that as the Decimals from 1 Day to 84, or 12 Weeks, are true to 7 Places; these for Quarters are accurate, tho' a Quarter is not a precise Number of Days, but 91 Days and $\frac{1}{4}$ of a Day; reckoning 365 Days to a Year: So that in applying this Table to the Calculations of Interest, for which chiefly it is designed,

you must reckon 91 Days to a Quarter; by which means the Decimal will be a little greater than what corresponds to 91 Days. But if we continue the Table, then the Decimals for 13 Weeks, 26 W. 39 W. 52 W. (which are less than

1 Year by 1 Day,) are as in the Margin. And if we reckon 13 Weeks a Quarter, then these are the Decimals for 1, 2, 3, 4 Quarters; but deficient for the exact Quarter of a Year, &c. And per-

haps it may be best to use these Numbers. So that if the time is within 13 Weeks, or 91 Days, take the Decimals in the 2 upper Parts of the Table; and if it exceeds 91 Days, take for 91 Days (= 13 W.) or 182 Days (= 26 W.) or 273 Days (= 39 W.) the Decimals in this last Part; taking Decimals for what Days are over any of these Numbers, and less than 91, in the former Part; and add all together. In short, Reduce the Number of Days to Quarters, Weeks, and Days, (at 91 Days to a Quarter,) and take their corresponding Decimals and add together. You shall see the Application particularly afterwards.

USE of the preceding TABLES.

I. To find the Decimal of any Integer (which is in the Tables) answering to any Number, simple or mix'd, less than that Integer.

Rule. (1.) If it is a simple Number, seek it in the Left Column of the Table relating to that Integer, and against it you have the Decimal sought: So for 7s. (in the Table of Money) we find .35 l.

(2.) If it is a Mix'd Number, seek the Decimals answering to the several Members; their Sum is the Decimal sought. *Examp.* The Decimal for 9s. 8d. is .483 (the 3 circulating for ever) which is the Sum of .45 and .0333f. the Decimals of 9s. and of 8d.

Observe. Decimal Tables would be more compleat, if they were made up for every Number, Mix'd as well as simple, less than the Integer; but as this would swell them to a great Bulk, so these for the simple Numbers are sufficient; because from them the others can be got easily as there is occasion. Or if any body wants such Tables, they are easily made, either by adding their Parts, or by the Method of *Case 2. Probl. 3.*

II. Having the Decimal of any Integer to find the corresponding Number, simple or mix'd, of known inferior Species.

Rule. (1.) Seek the given Decimal in the Table; if you find it there, against it stands the Number sought: So against .75 in the Table of Money stands 15s.

(2.) If the given Decimal is not exactly found in the Table, take the next lesser found there, the Number against it is Part of the Answer; then take the Difference betwixt that Decimal and the given one; and seek it or the next lesser in the Table, and against it you have another part of the Answer in a lower Species than the preceding part. Go on thus as long as you can, and you'll find the Answer as near as possible in known Species.

Examp. 1. To find the Value of .6875 l. I seek this in the Table, but the nearest to it (lesser) which I can find is .65, to which answers 13s. then the Difference of .6875 and .65 is .0375, which I find in the Table, against 9d. therefore the Answer is 13s. 9d.

Examp. 2. For .4768 l. the nearest lesser Decimal is .45 against 9s. then .4768 less .45 is .0268, and the next less than this is .025 against 6d. then .0268 less .025 is .0018, the next less than which is .00104, &c. against 1 farth. So the Answer is 9s. 6d. 1f. with a Fraction of a Farthing.

But *observe*, That if any Decimal given is not found exactly in the Tables, the Value of it may be had in most cases as easily, by Reduction (as in *Probl. 12. Reduction of Vulgar Fractions.*) And for any Integer in the preceding Tables, it will be sufficient to take the first three Places after the Point. But the easiest way to solve the Problem is by such compleat Tables as are already mentioned.

As the Decimals of Money are of the greatest use, so also there is an easy way of finding the Value of any Decimal of a Pound; or finding the Decimal for any Number less than 1 l. without Tables. Thus:

I. To find the Value of any Decimal of 1 l. without Tables or Pen.

Rule. Take the first three Figures after the Point, neglecting the rest; then double that Number which stands in the first Place (after the Point) it is so many Shillings of the Answer. And if the Figure in the second Place is 5 or greater, add 1 to the Shillings already found; then take what the Figure in the second Place exceeds 5, with the Figure in the third Place, (and if there is no Figure in the third Place, suppose 0) consider these two Figures, in order as they stand, as one Number. If they make a Number not exceeding 23, take so many Farthings (and reduce them to Pence) for the remaining Part of the Answer:

Answer: But if that Number exceeds 23, take 1 from it, and the Remainder is so many Farthings in the Answer. Thus you shall have the Answer exact in all its Value of known Species; or so, that the Error shall not be 1 Farthing.

Examp. 1. $.4l. = 8s.$

2. $.35l. = 7s.$

3. $.248l. = 4s. 11d. 3f.$

4. $.317l. = 6s. 4d.$

Examp 5. $.089l. = 1s. 9d. 2f.$

6. $.67l. = 13s. 5d.$

7. $.038l. = 9d. 1f.$

8. $.04l. = 9d. 3f.$

If you compare these *Examples* with the *Rule*, the manner of finding the Value will be plain, without any further Explication.

The *Reason* of this Rule is thus:

1. Since 1 Shilling is the $\frac{2}{20}$ Part of a Pound, and double any Number of 10th Parts, makes so many 20th Parts; (so $\frac{2}{10} = \frac{4}{20}$) therefore double the Figure in the 1st Place (whose Den^r is 10th Parts) is equal to so many 20th Parts, or Shillings. Again,

2. Since $\frac{1}{20} = \frac{5}{100}$, therefore 5 in the 2^d Place (whose Den^r is 100th Parts) is 1 Shilling. Then,

3. The Figure in the 3^d Place has 1000th Parts for its Den^r, and this with the Number over 5 in the 2^d Place, makes so many 1000th Parts; which is little less than so many Farthings; because 1 Farthing is $\frac{1}{240}$ Part of a Pound. But when we make up a Decimal Table for Farthings from 1 to 47, (which is 11d. 3f.) we find this true in Fact, That from 1 to 23 Farthings the Figures in the 2^d and 3^d Places of the Decimal are the same with the Number of Farthings: But from 24 to 47, the Figures in the 2^d and 3^d Places make a Number 1 more than the Number of Farthings. And tho' in all these Decimals (except that for 6d. or 24f.) there are Figures after the 3^d Place, yet their Value is not 1 Farthing, because they do not make .001, which is less than 1 Farthing.

II. To find the Decimal of a Pound, answering to any Number of Shillings, Pence, and Farthings, less than a Pound, without Tables or Pen.

Rule 1. If the Number of Shillings is even, take its Half and set in the first Place after the Point, (Ex. 1.) If it's odd, set the Half of the next lesser even Number in the first Place, and 5 in the 2^d; (Ex. 2.) then reduce the d. and f. to f. and if they are fewer than 24 (i. e. the d. fewer than 6 = 24f.) set that Number in the 2^d and 3^d Places (i. e. in the 3^d Place if it's but one Figure, (Ex. 3.) and if it has two, add that which is in the Place of Tens to the Figure standing already in the 2^d Place, if there is any, and set the other in the 3^d Place, Ex. 4.) But if these Farthings exceed 23 (i. e. if the d. exceed 5) add 1 to them, and set that Number in the 2^d and 3^d Places as before, (Ex. 5, 6.) Thus you have the Decimal sought, true to three Places, which is sufficient for common use. But,

2. If you would compleat the Decimal, then if the Number of f. to which the d. and f. in the Question are equal, do not exceed 23, take that Number of f. or, if they exceed 23, take the Remainder after 24 is subtracted from them, and divide that Number or Remainder decimally, (viz. by prefixing 0's to it) by 24 (which is easily and readily done by 4 and 6) the Quote, which will either be determinate, or circulate on 6 or 3, being set after the Figures already found, the Decimal is compleated.

Examp. 1. $4s. = .8l.$

2. $13s. = .65$

3. $6s. 2f. = .308$

4. $8s. 3d. = .4125$

5. $7s. 6d. = .375$

Examp. 6. $9s. 9d. = .4875$

7. $7d. 3f. = .0322916 \text{ \&c.}$

8. $4s. 8d. 3f. = .2364583 \text{ \&c.}$

9. $5s. 4d. 1f. = .2677083 \text{ \&c.}$

The
..

The Reason of this Rule is this:

1. For the Shillings: The half of any Number of 20th Parts (*i. e.* of any Number of Shillings) makes so many 10th Parts; and if there is an odd Shilling, it is equal to 5 in the 2^d Place, or $\frac{5}{100}$ Parts, because $\frac{1}{20}$ is $\frac{5}{100}$.

2. For the Pence and Farthings: If we make a Table of Decimals for any Number of Farthings from 1 to 47 (equal to 11 *d.* 3 *f.*) then for any Number less than 24 (or 6 *d.*) the Decimal has that Number in the 3^d, or 2^d and 3^d Places, (*i. e.* it will have so many 1000th Parts. And if these Farthings exceed 23, the Decimal has 1 more 1000th Parts. Again, For any Number of Farthings less than 24 (or 6 *d.*) consider, that because 1 Farthing is $\frac{1}{96}$ Part of a Pound, which is greater than $\frac{1}{1000}$ Part; therefore besides so many 1000th Parts, there must be added such a Decimal as is equal to the Difference of so many 960th Parts and 1000th Parts. Now, if we subtract $\frac{1}{1000}$ from $\frac{1}{96}$ the Remainder is $\frac{24}{96000}$; wherefore any Number less than 24, of 24000th Parts, will make a Decimal, whose first Place will fall in the fourth Place after the Point: Consequently this Decimal which remains to complete the Decimal sought, falls after the 3 Places already found.

Lastly, If the Number of Farthings exceeds 23, it is either 24; and then there is nothing to be added to the Number of 1000th Parts already set down: Or it's more than 24. And what's more, being less than 23; the Decimal to be added for that, comes under the same Rule as the last Article.

QUESTIONS, *showing the Application of Decimals in Multiplication and Division.*

Quest. 1. There is 14 *l.* : 8 *s.* : 6 *d.* in each of 6 Bags: How much is in the whole?
Answer, 86 *l.* : 11 *s.* Thus, 14 *l.* : 8 *s.* : 6 *d.* is 14.425 *l.* which multiplied by 6 produceth 86.55 equal to 86 *l.* : 11 *s.*

Quest. 2. If 1 Yard of Cloth cost 14 *s.* : 8 *d.* what is the Value of 24 *yds.* : 3 *qr.* : 1 *na.*
Ans. 18 *l.* : 3 *s.* : 2 *f.* nearest. Thus, 14 *s.* : 8 *d.* is .7333 *l.* and 24 *yds.* : 3 *qr.* : 1 *na.* is 24.7625 *yds.* Which multiplied by .7333 produces 18.158 *l.* which is 18 *l.* : 3 *s.* : 2 *f.* nearly.

Quest. 3. If 30 *l.* buy 124 *yds.* : 1 *qr.* : 2 *na.* of Cloth, How much will 1 *l.* buy?
Answer. 4 *yds.* : 2 *na.* and .32 nearly. Thus, 124 *yds.* : 1 *qr.* : 2 *na.* is 124.375, which divided by 30, quotes 4.145 *l.* which is 4 *yds.* : 2 *na.* and .32 nearly.

Quest. 4. If 8 *l.* : 9 *sh.* : 4 *d.* buy 3 hundred Weight : 1 *qr.* : and 18 Pound of Sugar, What may be bought for 1 *l.*? *Answer*. 1 *qr.* : 17 *lb.* and .1 nearly. Thus, 8 *l.* : 9 *sh.* : 4 *d.* is 8.4666 *l.* and 3 *C.* : 1 *qr.* : 18 *lb.* is 3.410714, which divided by the other, quotes .4028 *l.* which is 1 *qr.* : 17 *lb.* and .1 nearly.

Observe, The Use and Application of Decimals will more fully appear in the remaining Parts of this Work; especially by applying them in *Book 6.* As to which Application, this in general only needs to be further said here, That any Integer being consider'd as the highest Denomination, all Numbers or Quantities less than that are to be expressed decimally by taking the Decimal of that Integer answering to that lesser Quantity; and in the same Question using Decimals of the same Integer for all Numbers of the same kind, (*i. e.* for all Numbers of Money, use the same Integer as 1 *l.*) Then multiply and divide by these Numbers according to the Rules of Decimals. And in Multiplication you may use the manner of Contraction explained in *Ch. 8. §. 5.*

I must *observe* in the last place, that most Questions in common Business are sooner done without Decimals, by the common Methods of Reduction; but when to use Decimals, or the common Methods, must be left to every body's own choice: and indeed a good deal of Practice will be necessary to enable one to chuse judiciously. And particularly as to the preceding Examples, *observe*, That the first is easier done by the common Method; because it can be done without Reduction, the Multiplier being small. The second and fourth; cannot be done by any Method hitherto explain'd, (except by Decimals;) because by the Method of Reduction they require both Multiplication and Division; as you will afterwards understand in the *Rule of Three*, (*Book 6.*) The third may be easily done, the common way.

B O O K III.

Of the Powers and Roots of Numbers.

C H A P. I.

Containing the T H E O R Y.

D E F I N I T I O N S.

I. **A** *Power* is a Number, which is the Product of a certain Number of equal Factors; i. e. of the same Number multiply'd into itself continually a certain Number of Times.

II. A *Root* is a Number, by whose continual Multiplication into itself another Number (which is called the Power) is produced.

Example. Let any Number 2 be multiplied into itself, the Products are 4 ($= 2 \times 2$) 8 ($= 4 \times 2$) 16 ($= 8 \times 2$) &c. Then is 2 called the Root of these Products, which are called the Powers of that Root.

Hence it is plain, that Power and Root are relative things. Every Power is the Power of some Root, and every Root is the Root of some Power: So that by calling one Number the Power or Root of another, we mean that it is the Number produced, or the Number producing that other by continual Multiplication.

But the several Powers, and the Root in relation to these, are also distinguished by particular Names, which shall next be explained.

III. The first Product (*viz.* that of the Root multiplied by itself) is called the Square of the Number multiplied; which in respect of the other is called the Square Root. So 4 is the Square of 2, and 2 the Square Root of 4, because $2 \times 2 = 4$.

The second Product (*viz.* of the Square by the Root) is called the Cube, and the Root in respect of it is called the Cube Root: So 8 is the Cube of 2, and 2 is the Cube Root of 8; for $2 \times 2 \times 2$, or $4 \times 2 = 8$.

Others of the Powers had also particular Names among the Ancients; but they are very complex and burthensome to the Memory, and tend no way to the Improvement or Easiness of the Science: Whereas it is obvious that we have no more to do, but distinguish them by their Order in the Series of Products, calling the first Product the first Power, the second Product the second Power, and so on; whereby these Names do of themselves in a very simple and easy manner distinguish the several Powers, in consequence of the general Definition of a Power: for they express the Number of Multiplications of the

Root in the Production of each Power; which the ancient Names do not. For the Names Square and Cube, of which the rest were compounded, are Names of geometrical Quantities applied to Numbers, only from this Consideration, that the Measures of these Quantities are found by such an Application of Numbers, as do produce the Numbers, which are hence called *Square* and *Cube*.

But observe again, that tho' in consequence of the preceding Definitions of Power and Root, these Terms ought always to be contradistinguished, so that the Products only can be called Powers; yet for the sake of a particular Conveniency, which we shall presently understand, the Root is called the first Power, and the Products in order are called the second, third, &c. Power, as here:

2.	2 × 2.	4 × 2.	8 × 2.	16 × 2.	&c.
2.	4	8	16	32	&c.
Root or 1st Power.	Square or 2d Power.	Cube or 3d Power.	4th Power.	5th Power.	

In which Method the Root is the same with the first Power, and contradistinguished only from the superior Powers, with respect to which we call it the second or third, &c. Root; tho' more commonly we use the Names Square and Cube, and Square Root, Cube Root; using the Names fourth, fifth, &c. Power and Root, for the degrees above the Cube or third Degree.

Of the universal Notation of Powers and Roots.

I. Of POWERS.

TAKE any Number A for a Root, and the Series of its Powers according to the Definition will be thus:

A.	AA.	AAA.	AAAA.	&c.
Root, or 1st Power.	2d Power.	3d Power.	4th Power.	

Each of these Terms expressing the continual Product of A, taken so oft as it is placed in each of them, which being once more at every Step gradually from the Root, we have also this more convenient Method of expressing them, viz. by writing only the Root with a Number over it, to signify how oft the Root is to be taken, or placed as a Factor in producing that Power. Thus the 4th Power of A is AAAA, to be written, according to this other Method, thus, A^4 ; and so of others, the whole Series of the Powers being represented thus:

$$A^1, A^2, A^3, A^4, A^5, A^6, \&c.$$

When a Number A has no Figure or Mark of Power, it's supposed to be the first, so that A^1 or A are equivalent.

These Figures we call *Indexes* or *Exponents* of the Powers; because by shewing the Number of Factors, they shew what Power is signified by that Expression, or what Term in order of the Series; for the Numbers of Factors increase gradually in the Series, the Root

Root standing alone in the 1st Term, twice in the 2^d, and so on. And since the Denominations of the Powers are taken from their Places in the Series, they do also express the Number of equal Factors, or the Number of times the Root is placed by Multiplication in every Power, and consequently the Index is the Denomination; so if $A=2$ then $A^3=AAA=2\times 2\times 2=8$.

Again: By this Method any Power indefinitely may be expressed by a general or indefinite Index thus, A^n ; which is any Power of A , according to the Value we put upon the Index n .

Hence any Series of Powers decreasing from a given one down to the Root may be expressed thus:

$$A^n, A^{n-1}, A^{n-2}, A^{n-3}, A^{n-4}, \&c.$$

Still subtracting one more from the Index till it become equal to 1, and then you have the Root.

II. For Roots.

The Root of any Number considered as a Power may also be conveniently expressed by that Number with an Index; thus, over the Number which is the Denomination of the Root, set $\sqrt{}$, in form of a Fraction; this is the Index of the Root: For Example; The Square Root of A is $A^{\frac{1}{2}}$, the Cube Root $A^{\frac{1}{3}}$, the 4th Root $A^{\frac{1}{4}}$, and so on; so that if $A=4$, then $A^{\frac{1}{2}}=2$; Or if $A=8$, then $A^{\frac{1}{3}}=2$; And an indefinite Root thus, $A^{\frac{1}{n}}$.

There is also another way of marking Roots by this Mark $\sqrt{}$, setting the Power before it, and the Index above it: Thus the Square Root of A is $\sqrt[2]{A}$, the n Root is $\sqrt[n]{A}$.

And now, to understand the Conveniency of distinguishing the Powers by their Order in the Series, *i. e.* by the Number of Factors or Indexes, Consider that the various Powers of the same Root differ only by these Indexes, or Numbers of Factors; and the Rules for their mutual Application to one another by Multiplication and Division, (by which chiefly their different Properties are discovered,) depending upon the Consideration of these different Numbers of Factors, it is a more simple and easy Method to make the same Number express both the Number of Factors, and give a Denomination to the Power; which would not be, if we should begin the Numeration of the Powers at the first Product, calling AA , or A^2 the First Power. It is true indeed, that by this Method the Denomination would always be one less than the Index or Number of Factors, and so would be a certain regular Method of shewing that Number; but still the other is more simple and easy: Which the Applications to be made afterwards will make appear more evidently.

There is one thing more you may observe upon this Method of denominating Powers, *viz.* That tho' the Root is not a Product of itself multiplied into itself, and so is not a Power according to the general Definition; yet we may always contra-distinguish Root and Power, understanding them according to the general Definition, and at the same time take the Denominations of Powers from the Indexes or Numbers of Factors; provided we understand these Denominations or Indexes to express no other thing but the Number of Factors, *i. e.* a Power composed of so many Factors as the Index expresses, and not as signifying the Degree and Order of the several Powers from the first Product, which, according to the general Definition, is the first Power, tho' the Index is 2; so for Example, A^4 is called the Fourth Power, not as being the fourth Term in the order of Products, for it is only the 3^d Product, but as being composed of four Factors; *viz.* the Root stated as a Factor four times; so $A^4=AXAXAXA$.

But now after all, it's to the same Purpose in which of these Views you take the Denomination; for the whole Conveniency lies in having the Number of Factors expressed, which is done either way. Others again consider 1 as a Factor in every Power, and then they

they make the Index express the Number of Multiplications by which a Power is produced; Thus $A^2 = 1 \times A \times A$; in which are two Multiplications, 1st $1 \times A$, 2d $1 \times A$ by A .

DEFIN. IV. Powers or Roots are called *Like* or *Similar* to one another, whose Denominations or Indexes are the same; so A^2 , B^2 , or A^3 , B^3 , are similar Powers, and their similar Roots, $A^{\frac{1}{2}}$, $B^{\frac{1}{2}}$. Such are also said to be Powers or Roots of the same Degree or Order.

And when the Indexes are different or unequal, they are called *unlike* or of a different Order: As A^2 , B^3 ; also $A^{\frac{1}{2}}$, $B^{\frac{1}{3}}$.

V. The finding any Power of a Number is called *Raising* that Number to such a Power; as finding the 4th Power of A is called Raising A to the 4th Power: And this is also call'd in general, *Involving*, or the *Involution* of that Number, according to the Index of the Power.

VI. The finding any proposed Root of a Number, is called the *Extracting* of such a Root from that Number; as finding the Cube Root of A is call'd the extracting the Cube Root of A ; and this we call in general *Evolving*, or the *Evolution* of that Number, according to the Index of the Root.

VII. As any Number may be made a Root, and involved to any Power, so if a Number C is a Power of another B , which is again a Power of another A , then may C be called a *Compound Power* of A , i. e. a Power of a Power of A , (as with respect to B it's a simple Power,) and may be generally expressed thus: $\overline{A^m}^n$, that is, the n Power of A^m . Example: 64 is the Square of the Cube of 2, for it is the Square of 8, which is the Cube of 2. The Composition may also consist of more than two Members, as the n Power of the m Power of A^o .

VIII. If any Number A is a certain Root of another B , which is also a certain Root of another C , then may A be called a *compound Root* of C (as with respect to B it is a simple Root) and may be expressed thus, $\overline{C^{\frac{1}{m}}}^{\frac{1}{n}}$, that is, the n Root of $C^{\frac{1}{m}}$. Example: 2 is the Cube Root of the Square Root of 64.

SCHOLIUM. That these Compound Powers and Roots must be equal to some immediate or simple Power or Root of the Number to which they are referred, will easily be understood from the Nature of such Numbers; that is, the n Power of the m Power of A is some immediate simple Power of A , as A^o ; and so of Roots: How such simple Expressions are found, shall be explained in its place.

IX. A Number which is first considered as a certain Root of another, as the n Root, may be itself involved according to some other Index m , and this Power being referred to the same Number to which the preceeding Root was referred, may be called a *mix'd Power* of that Number, so if $B = A^{\frac{1}{n}}$, then the m Power of B , that is, the m Power of the n Root of A is a mix'd Power of A (which referred to B is a simple Power) and may be expressed $A^{\frac{m}{n}}$. Example: 9 is the Square of the Cube Root of 27, for it is the Square of 3, whose Cube is 27.

In the same manner, a Number being considered as a certain Root of a certain Power (whose Index is different from that of the Root) of a Number, it may be called a *mix'd Root*, as the m Root of the n Power of A , represented thus: $\overline{A^n}^{\frac{1}{m}}$. Example: 9 is the Cube Root of the Square of 27, for the Square of 27 is 729, which is also the Cube of 9.

Observe,

Observe, For either of these Kinds, *viz.* a mix'd Power, or mix'd Root, we may institute this manner of Representation $A^{\frac{n}{m}}$, which may signify either the m Root of A^n , or the n Power of $A^{\frac{1}{m}}$. But then observe, that we can't make it represent either of these indifferently, till we have first demonstrated that they are equal; which shall be afterwards done; and till then, I shall only use it for the m Root of the n Power.

SCHOLIUM. Every Number is a Root of any Order whatever, because it may be involved to any Power; but every Number is not a Power of any Order; some being Powers of no Order but the first, which is only being a Root; *i. e.* there are some Numbers which cannot be produced by the continual Multiplication of any Number whatever; and such are 3, 5, 6, 7, and an infinite Number of others. Some again are Powers of one particular Order only; as 4, which is only a Square; and 8, which is only a Cube. Some, in the last place, are Powers of more than one Order, but limited to a certain Number of different Orders; as 64 is both a Square and a Cube; its Square Root being 8, and the Cube Root 4; for $8 \times 8 = 4 \times 4 \times 4 = 64$: The Demonstration of these things you'll learn afterwards; to mention them in general is enough here, which was only necessary for the sake of the following Definition.

DEFIN. X. When a Number A is proposed as a Power of any Order n , and yet is not a Power of that Order, *i. e.* if it has no determinate Root of that Order, or there is no Number which involved as the Index n directs, will produce that Number; yet it has what we may call an indeterminate Root, (as shall be afterwards explain'd) and this imagin'd Root, under the Notion of a true and compleat Root, is called a *Surd* (*i. e.* inexpressible) Root, and is represented in the general Form $A^{\frac{1}{n}}$; and such Roots as are real, are, in Distinction from Surds, called *Rational Roots*. *Exam.* 8 is not a Square; for there is no Number, which multiplied into itself, will produce 8. No Integer will, since $2 \times 2 = 4$, and $3 \times 3 = 9$; and that no mix'd Number betwixt 2 and 3 can do it, will be afterwards demonstrated.

But now as to Surds, don't mistake, as if such Roots or Representations were nothing at all, or so merely imaginary as to be of no Use in Arithmetick; for though there be no such determinate or assignable Number, whose n Power is equal to A ; yet we can find Numbers mix'd of Integers and Fractions, that shall approach nearer and nearer to the Condition required, *in infinitum*; *i. e.* we can find a Series of Numbers decreasing continually, whose Sum taken at every Step is a Number, the Power of which approaches nearer and nearer to the given Number; and this Series consider'd in its infinite Nature, as going on by the same Tenor and Law without end, and thereby approaching infinitely near to the Condition of a true Root, is truly and properly what we call a *Surd*; which, 'tis plain, is something real in itself, though we can't express the whole Value of it by any definite Number; for that is contrary to its Nature: So we find that the *Surd* Roots of different Numbers have certain Connections and relative Properties the same way as rational Numbers have; (all which things shall be demonstrated in their proper place.)

Therefore we conceive Surds as Quantities compleat of their own kind, and so use the same general Notation for Surds and rational Roots: And hence the following Theory relating to Roots are to be understood generally, whether they are Surds, or rational Roots; concerning the Reason and Application of which to Surds, you'll learn more Particulars afterwards in *Chap. 3*.

XI. The Powers and Roots of *Fractions* are to be understood the same way as of whole Numbers; that is, any Fraction being continually multiplied into itself, is a *Root* or *first Power*, with respect to the Products which are the superior Powers.

Exam-

Example. $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$, and $\frac{4}{9}$ is therefore the Square of $\frac{2}{3}$, and may be universally expressed thus, $\frac{a^2}{b^2}$; or thus, $\frac{a^2}{b^2}$; and the Root thus, $\sqrt[n]{\frac{a^2}{b^2}}$; or thus $\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}$.

Observe, That only is the proper and immediate Power of a Fraction, whose Terms are the Powers of the Terms of that Fraction; yet as the same Fraction may be expressed in various Terms, so all equivalent Fractions may be taken for the Power or Root of the same Fraction, because they have the same Effect in all Operations, if any one of them is $\frac{4}{9}$, according to the Definition. Thus, because $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$, and because $\frac{4}{9} = \frac{2}{3}$, therefore $\frac{4}{9}$, which is the Square of $\frac{2}{3}$, may be also called the Square of $\frac{4}{9}$, also $\frac{8}{18}$ ($= \frac{4}{9}$) may be called the Square of $\frac{2}{3}$ or $\frac{4}{9}$.

SCHOLIUM. When a complex literal Expression is consider'd as a Root, to express any Power of it we draw a Line over the whole Expression, and then annex the Index; thus, $\overline{A+B}^n$ is the n Power of $A+B$, and \overline{AB}^n is the n Power of the Product $A \times B$: But if there is no Line over, then the Index is applied only to the Member over which it is immediately set, as $A+B^n$ is only the Sum of A and B^n , and AB^n the Product of $A \times B^n$.

A X I O M S.

Ist, LIKE Powers or Roots of equal Numbers (however differently expressed) are equal. Thus: If $A=B$, then $A^n=B^n$, and $A^{\frac{1}{n}}=B^{\frac{1}{n}}$. But unequal Numbers have their similar Powers and Roots unequal; the greater having the greater Number for its Power or Root. Hence,

COROL. If the different Powers of unequal Numbers are equal, the Power of the lesser Root has the greater Index. Thus: If $A^n=B^m$, and A be less than B ; then is n greater than m : For if $n=m$, then is A^n less than B^m ; and much more is it so, if n is less than m . *Exam.* $8^2=4^3=6^4$.

II^d, If a Number is involved to any Power, and from this Power a Root of the same Denomination is extracted, this Root is the same Number which was first involved. So if A is involved to the n Power, and of this Power, *viz.* A^n , we extract the n Root, it is equal to A , *i. e.* $A = \sqrt[n]{A^n}$. And reversly extract any Root of a Number, and then involve that Root to a Power of the same Denomination; this Power is the same Number from which the Root was first extracted; so the n Power of $A^{\frac{1}{n}}$, or $\overline{A^{\frac{1}{n}}}^n = A$. Hence,

COROL. I. An Expression of this Sort $A^{\frac{n}{n}}$, where the Numerator and Denominator of the Index are equal, whether it is understood as the n Root of the n Power; or the n Power of the n Root of A , is no other thing in Effect and Value but A . Hence again;

2. Involution and Evolution are directly opposite, the one undoing the Effect of the other; whereby they are mutual Proofs one of the other.

T H E O R E M I.

IF two similar Powers of different Numbers or Roots are multiplied together, the Product is the like Power of the Product of these Numbers or Roots. Thus, the Product of two Squares is the Square of the Product of their Roots. *Universally* $A^n \times B^n = \overline{AB}^n$.

Demonstr

$$\begin{array}{l} A=2, A^2=4 \\ B=5, B^2=25 \\ \hline AB=10. A^2 \times B^2 = \\ 100 = \overline{AB^2}. \end{array}$$

Demonstr. In the Product $A^n \times B^n$, A and B are applied as Factors equal times; so that in the whole Product there is a Number of Factors equal to $2n$; and consequently the Product AB, consider'd as one Number or Factor, is applied in that Product n times [for in whatever Order the Factors are taken, the Product is still the same]; so that the Product given is AB raised to the n Power, or $\overline{AB^n}$. Thus particularly, $A^3 \times B^3 = AAA \times BBB = AB \times AB \times AB = \overline{AB^3}$.

COROLL. Hence we learn, how the Product of a Number expressed as a Power, and another not expressed as such (but supposed to be one) may be reduced to such an Expression. Thus, $A^n \times B = \overline{A \times B^{\frac{1}{n}}}$; for the n Power of A is A^n , and the n Power of $B^{\frac{1}{n}}$ is B, therefore $A^n \times B = \overline{A \times B^{\frac{1}{n}}}$.

SCHOL. This Theorem is true also, when the Roots are the same; as $A^n \times A^n = \overline{AA^n}$. But I have not taken this Case into the Theorem, because it falls within another (see Theor. 6.) where the Product is expressed in a more simple and convenient way. You are to understand the same of the Theorems 2, 3, and 4.

THEOREM II.

IF two similar Powers of different Numbers or Roots are divided one by the other, the Quote is the like Power of the Quote of the given Roots. Thus the Quote of two Squares is the Square of the Quote of their Roots *universally* $B^n \div A^n = \overline{B \div A}^n$.

$$\begin{array}{l} A=3. A^3=27 \\ B=6. B^3=216 \\ \hline B \div A=2. B^3 \div A^3=8 \\ \overline{B \div A}^3=8. \end{array}$$

Demonstr. 1. Suppose A, B are Integers, then $B \div A$ expressed fractionally is $\frac{B}{A}$, whose n Power, according to *Defin.* 11. is

$$\frac{B^n}{A^n} = B^n \div A^n.$$

2. Suppose B and A fractional; thus, $B = \frac{c}{o}$ and $A = \frac{d}{a}$, then is $B^n = \frac{c^n}{o^n}$ and $A^n = \frac{d^n}{a^n}$.

Also $B^n \div A^n = \frac{c^n}{o^n} \div \frac{d^n}{a^n} = \frac{c^n a^n}{o^n d^n} = \frac{\overline{ca}^n}{o d^n}$ (*Theor.* 1.) But $\frac{ac}{od} = \frac{c}{o} \div \frac{d}{a} = B \div A$, and its n

Power is $\frac{\overline{ac}^n}{o d^n}$, which is $\overline{B \div A}^n$.

SCHOLIUM. I have not here considered, whether $B^n \div A^n$ is a whole Number or a Fraction; for which so ever of them it be, you see plainly that it is equal to $\overline{B \div A}^n$. In another place you'll find it demonstrated, that according as $B^n \div A^n$ is integral or fractional, so is $B \div A$; and reverfly.

The following Corollaries 2, 3, 4, and 5, are deduced from this and the first Theorem jointly considered.

COROLL. 1. The Quote of two Numbers, whereof one is expressed as a Power and the other not, (tho' supposed to be one,) may be reduced to such an Expression

thus; $A^n \div B = \overline{A \div B^{\frac{1}{n}}}$; for A^n is the n Power of A, and B is the n Power of $B^{\frac{1}{n}}$.

2. If any Product and one of the Factors are similar Powers, the other is also a similar Power, and its Root is the Quote of the Roots of the other two Terms. Thus, if

A^n

$A^n \times B = D^n$, then is $B = D^n \div A^n = \overline{D \div A}^n$ whose n Root is $D \div A$. In another Place it shall be demonstrated that $D \div A$ must be integral if $D^n \div A^n$ or B is so.

III. If the Dividend and either of the other two Terms, *viz.* Divisor and Quote, are similar Powers, the other of those Terms is also a similar Power, whose Root is the Product of the Roots of the Dividend and the former of the other two Terms. Thus if

$B^n \div A = D^n$, then $A = B^n \times D^n = \overline{B D}^n$.

4. The Product of two Numbers being a Power, the Factors are either both like Powers with the Product, or neither of them is so. Also, if one Factor is a Power, and the other not a like Power, the Product is not a like Power. Let $A B = p^n$, if A is a Power of the Order n , so is B , for if $A = a^n$ then is $a^n \times B = p^n$, and consequently B is of the Order n (by the 2d) whence also if the one A or B is not a Power of the Order n , neither is the other; for one being so, the other would be so too. Again, if $A^n \times B = D$, and B not a Power of the Order n , neither is D ; for if D were so, B would be so too.

5. If the Quote of two Numbers is a Power, these Numbers are either both Powers like the Quote, or neither of them is so: This is the Reverse of the last.

SCHOLIUM. In these Corollaries you are to understand Rational Powers; for otherwise any Number may be represented as a Power of any Order.

THEOREM III.

If two similar Roots of different Numbers are multiplied, the Product is the like Root of the Product of these Numbers. Thus two Cube Roots produce a Number which is the Cube Root of the Product of their Cubes. Universally, $A^{\frac{1}{n}} \times B^{\frac{1}{n}} = \overline{A B}^{\frac{1}{n}}$.

Example:

$$\begin{aligned} A &= 27. A^{\frac{1}{3}} = 3 \\ B &= 8. B^{\frac{1}{3}} = 2 \\ AB &= 216. A^{\frac{1}{3}} \times B^{\frac{1}{3}} = 6 \\ \overline{AB}^{\frac{1}{3}} &= 6 \end{aligned}$$

Demonstration This is but the Reverse of the 1st Theorem, and follows easily from it. Thus, Let $A B$ be any similar Powers, then is $A \times B = \overline{A^{\frac{1}{n}} \times B^{\frac{1}{n}}}^n$ by Theor. I. and by Ax. 1. $\overline{A \times B}^{\frac{1}{n}} = A^{\frac{1}{n}} \times B^{\frac{1}{n}}$, which is the n Root of $A^{\frac{1}{n}} \times B^{\frac{1}{n}}$.

COROLL. The Product of two Numbers, whereof one is expressed as a Root and the other not, may be reduced to such an Expression thus; $A^{\frac{1}{n}} \times B = \overline{A \times B^n}^{\frac{1}{n}}$; for A is the n Power of $A^{\frac{1}{n}}$, and B^n of B ; therefore, by this Theorem $A^{\frac{1}{n}} \times B = \overline{A \times B^n}^{\frac{1}{n}}$. This is also the Reverse of Coroll. Theor. 1.

THEOREM IV.

If two similar Roots of different Numbers are divided, one by the other, the Quote is the like Root of the Quote of the one Number divided by the other. Thus, two Cube Roots give for a Quote the Cube Root of the Quote of the Cubes. Universally, $D^{\frac{1}{n}} \div A^{\frac{1}{n}} = \overline{D \div A}^{\frac{1}{n}}$.

Example:

$$\begin{aligned} D &= 216. D^{\frac{1}{3}} = 6 \\ A &= 27. A^{\frac{1}{3}} = 3 \\ D \div A &= 8. D^{\frac{1}{3}} \div A^{\frac{1}{3}} = 2 \\ \overline{D \div A}^{\frac{1}{3}} &= 2 \end{aligned}$$

Demonstr. 1. Suppose $D^{\frac{1}{n}}, A^{\frac{1}{n}}$ are both Integers, then are D, A also Integers, (*viz.* the Powers, or Products of integral Factors,) therefore $D^{\frac{1}{n}} \div A^{\frac{1}{n}}$ or $\frac{D^{\frac{1}{n}}}{A^{\frac{1}{n}}}$ is a Fraction in Terms

whose n Power is $\frac{D}{A}$; *i. e.* the n Root of $\frac{D}{A}$ is $\frac{D^{\frac{1}{n}}}{A^{\frac{1}{n}}}$ or $D^{\frac{1}{n}} \div A^{\frac{1}{n}}$.

2. If $D^{\frac{1}{n}}, A^{\frac{1}{n}}$ are fractional, then are D, A also fractional, (according to a former Observation.) Suppose $D^{\frac{1}{n}} = \frac{b}{a}$, and $A^{\frac{1}{n}} = \frac{d}{c}$, then is $D = \frac{b^n}{a^n}$, and $A = \frac{d^n}{c^n}$ (Ax. 1.) Also

$$D^{\frac{1}{n}} \div A^{\frac{1}{n}} = \frac{b}{a} \div \frac{d}{c} = \frac{bc}{ad}; \text{ and } D \div A = \frac{b^n}{a^n} \div \frac{d^n}{c^n} = \frac{b^n c^n}{a^n d^n} = \frac{\overline{bc}^n}{ad}; \text{ therefore } \overline{D \div A}^{\frac{1}{n}} = \frac{bc}{ad} = D^{\frac{1}{n}} \div A^{\frac{1}{n}}.$$

COROLL. The Quote of two Numbers whereof one is expressed as a Root, and the other not, may be reduced to such, thus; $D^{\frac{1}{n}} \div A = \overline{D \div A^n}^{\frac{1}{n}}$. For $D^{\frac{1}{n}}$ is the n Root of D , as A is of A^n .

SCHOLIUM. From the two last Theorems jointly consider'd, we have four Corollaries after the same manner as the 2d, 3d, 4th, 5th Corollaries, deduced from Theorem II'd and I'st.

THEOREM V.

THE Product or Quote of any like Mixt Power (or Root) of two different Numbers, is the like Mixt Power of the Product or Quote of these two Numbers: Thus

$$A^{\frac{n}{m}} \times B^{\frac{n}{m}} = \overline{AB}^{\frac{n}{m}} \text{ and } A^{\frac{n}{m}} \div B^{\frac{n}{m}} = \overline{A \div B}^{\frac{n}{m}}.$$

Demonstr. 1. $A^n \times B^n = \overline{AB}^n$ (Theor. I.) and $\overline{A^n}^{\frac{1}{m}} \times \overline{B^n}^{\frac{1}{m}} = \overline{A^n \times B^n}^{\frac{1}{m}}$ (Theor. III.) But $\overline{A^n}^{\frac{1}{m}} = A^{\frac{n}{m}}$ and $\overline{B^n}^{\frac{1}{m}} = B^{\frac{n}{m}}$ by the Notation; and since $A^n \times B^n = \overline{AB}^n$, hence $A^{\frac{n}{m}} \times B^{\frac{n}{m}} = \overline{AB}^{\frac{n}{m}}$.

2. $A^n \div B^n = \overline{A \div B}^n$ (Theor. II.) And $\overline{A^n}^{\frac{1}{m}} \div \overline{B^n}^{\frac{1}{m}} = \overline{A^n \div B^n}^{\frac{1}{m}}$, (Theor. IV.) = $\overline{A \div B}^{\frac{n}{m}}$, (Ax. I.) That is, $A^{\frac{n}{m}} \div B^{\frac{n}{m}} = \overline{A \div B}^{\frac{n}{m}}$.

Example. The Square Root of 16 is 4, and the Cube of 4 is 64, therefore the Cube of the Square Root of 16, or $16^{\frac{3}{2}}$ is = 64. In like manner, the Cube of the Square Root of 81, or $81^{\frac{3}{2}}$ is = 729. Then $81 \times 16 = 1296$, whose Square Root is 36, and the Cube of this is 46656 = 64×729 ; that is, $16^{\frac{3}{2}} \times 81^{\frac{3}{2}} = \overline{16 \times 81}^{\frac{3}{2}}$.

THEOREM VI.

IF two Powers of the same Root are multiplied, the Product is such a Power of the same Root, whose Index is the Sum of the Indexes of the Factors. Thus, The Product of the 2d and 3d Powers of any Number, is the 5th Power of the same Number. Universally, $A^n \times A^m = A^{n+m}$.

Example.

$$\begin{array}{l} A = 3. \quad A^2 = 9 \\ A^3 = 27. \quad A^2 \times A^3 = 243 \\ A^5 = 243. \end{array}$$

Demonstr. A^n and A^m being each a Product of A continually multiplied by itself, their Product must be a Product of A continually by it self; i. e. a Power of A .
U Also

Also it's plain, that in $A^n \times A^m$, the Root A is applied as a Factor as oft as the Sum $n+m$, so that $A^n \times A^m = A^{n+m}$. Particularly, suppose $A^2 \times A^3$, the Product is $A A \times A A A = A A A A A = A^5$.

SCHOLIUM. What's here proved for two Factors, holds equally for three or more: Thus $A^n \times A^m \times A^r = A^{n+m+r}$.

COROLL. Hence we learn how to find any Power of a given Root, without finding all the intermediate Powers; viz. by multiplying together two or more Powers of that Root, the Sum of whose Indexes is the Index of the Power sought: Thus, having the 3^d and 4th Powers, their Product is the 7th Power.

THEOREM VII.

IF one Power is divided by another Power of the same Root; the Quote is equal either to another Power of the same Root, or to a fractional Power whose Numerator is 1, and the Denominator some Power of the same Root. Thus, particularly, If the Dividend is greater than the Divisor, the Quote is a Power of the same Root whose Index is the Difference of the Indexes of the proposed Powers; and if the Dividend is less, the Quote is a fractional Power whose Numerator is 1, and the Denominator is such a Power of the given Root whose Index is the Difference of the given Indexes.

Thus $A^n \div A^m$ is either A^{n-m} , or $\frac{1}{A^{m-n}}$.

Example 1.
 $A=3$ $A^4=81$
 $A^2=9$ $A^4 \div A^2=9$
 $A^{4-2}=A^2=9$

Example 2.
 $A^2=9$ $A^5=243$
 $A^2 \div A^5=9 \div 243 =$
 $\frac{1}{A^3} = \frac{1}{A^3} = \frac{1}{81}$

Demonstr. This is the Reverse of the last Theorem, and the Reason of it contain'd in that, from the reciprocal Nature of Multiplication and Di-

vision, with that of Addition and Subtraction. Or, this subtracting of the one Index from the other, is only the taking out equal Factors from the Divisor and Dividend, i.e. dividing them equally; which makes the Quotes still the same: Thus, $A^5 \div A^2 = A^3 \div 1 = A^3$, and $A^2 \div A^5 = 1 \div A^3$.

THEOREM VIII.

EVERY Compound Power (or Power of a Power) of any Root is equal to such a Simple Power of the same Root whose Index is the Product of the given Indexes: Thus the 3^d Power of the 2^d Power is the 6th Power. Universally, the n Power of the m Power of A , or A^{m^n} is $= A^{nm}$.

Example. $A=2$. $A^2=4$
 $A^6=64=A^3$
Demonstr. $A^m \times A^m = A^{2m}$ (Theor. 6.) and involving or multiplying A^m by itself once more, the Index of the Product contains m once more; so that however oft A^m is employed as a Factor, as n times, the Index of the Product will be so many times m , or $m \times n$: But A^m employed as a Factor n times makes the n Power of A^m ; which is therefore equal to the mn Power of A , or A^{mn} .

SCHOLIUMS.

1. The same Reasoning is good, when the Composition consists of more than two Steps: Thus the n Power of the m Power of the r Power, is the mnr Power of A , or A^{nmr} .

2. If the Index of any Power is the Product of two Numbers, it may be considered as a Compound Power; or if it is the Product of more than two Factors, it may be reduced to a Compound of two, by taking any one of these Factors for one Member, and the Product of all the rest for the other. Thus, A^{nmr} is the r Power of A^{nm} , or the n Power of A^{mr} , or the m Power of A^{nr} . In short, if the Index of a Power is the Product of

of other Numbers, whatever Variety is in the Composition of that Product, there is the same Variety in the Composition of that Power.

C O R O L L A R I E S.

1. Here we learn to find a Number, which is a Power of as many different Orders as can be proposed, *viz.* by multiplying the Indexes of all these proposed Orders continually into one another, and raising any Number to a Power, whose Index is that Product. Thus: To find a Number which is both a Square and Cube, raise any Number to the sixth Power. Universally, to find a Number which is a Power of the Orders n, m, r , raise any Number A to the $n m r$ Power, and A^{nmr} is the Number sought: for it is the r Power of A^{nm} , the n Power of A^{mr} , the m Power of A^{nr} .

Observe also, That if one of the Indexes is an *aliquot* Part of another, we need not multiply that Index which is the *aliquot* Part. Thus: To find a Number which is both a second and fourth Power, we need only to take some fourth Power, since every fourth Power is also a second Power. For because $2 \times 2 = 4$, therefore A^4 is the second Power of A^2 . By this you'll understand any other Case.

2. If out of a certain Power of a given Number, a Root is to be extracted of a different Index, and if the Name of the Root is an *aliquot* Part of the Name or Index of the Power; by dividing the Name of the Power by that of the Root, and applying the Quote as an Index to the given Number, we have an Expression for the Root sought. Thus; the Square Root of A^6 is A^3 , because $6 \div 2 = 3$. And universally, let $n \div m = r$. Then

is $A^{\frac{n}{m}} = A^r$, *i. e.* where there is a mixt or fractional Index (which expresses a certain Root named by the Denominator, of a certain Power named by the Numerator) if the Denominator is an *aliquot* Part of the Numerator; then dividing the Numerator by the Denominator, the Quote is an Index, which applied to the same Number, expresses the Value of that mixt Root.

T H E O R E M IX.

EVERY Compound Root (or Root of a Root) of any Number, is equal to such a simple Root of the same Number, whose Index is the Product of the proposed Indexes. Thus; the Square Root of the Cube Root is the sixth Root. Universally, the n Root of the m Root is the $n m$ Root, or $\sqrt[n]{\sqrt[m]{A}} = A^{\frac{1}{nm}}$.

Example: $A = 64$, $A^{\frac{1}{2}} = 8$, $A^{\frac{1}{3}} = 4$, $A^{\frac{1}{6}} = 2 = A^{\frac{1}{2} \cdot \frac{1}{3}}$. *Demonstr.* The Reason of this is contained in the preceding, because extracting of Roots is opposite to raising Powers. Or it may be demonstrated thus: Suppose $A^{\frac{1}{nm}} = B$, then $A = B^{nm}$ (*Ax. 1, 2.* for A is the $n m$ Power of its $n m$ Root) and $A^{\frac{1}{m}} = B^{\frac{n}{m}} = B^n$ (*Cor. 2. Theor. 8.*) Again, $\sqrt[n]{A^{\frac{1}{m}}} = B$ (*Ax. 1.*) But $B = A^{\frac{1}{nm}}$; therefore $\sqrt[n]{A^{\frac{1}{m}}} = A^{\frac{1}{nm}}$.

T H E O R E M X.

ANY Power of any Root of a Number is equal to the same Root of the same Power of that Number. Thus, the Square Root of the Cube of any Number is the Cube of the Square Root of that Number. Universally, $\sqrt[n]{A^m} = A^{\frac{m}{n}}$.

Example: $A = 9$, $A^{\frac{1}{2}} = 3$, $A^3 = 729$, $\sqrt[3]{A^{\frac{1}{2}}} = 27 = A^{\frac{1}{2} \cdot \frac{1}{3}}$. *Demonstr.* Suppose $A^{\frac{1}{m}} = B$, then is $A = B^m$ (*Ax. 1.*) and $A^n = B^{mn}$ (*Ax. 1. with Theor. 7.*) Again, $\sqrt[n]{A^{\frac{1}{m}}} = B^{\frac{n}{m}}$ (*Ax. 1.*) $= B^n$ (*Cor. 2. Theor. 8.*) But since $A^{\frac{1}{m}} = B$, therefore $\sqrt[n]{A^{\frac{1}{m}}} = B^n$; consequently $\sqrt[n]{A^{\frac{1}{m}}} = A^{\frac{n}{m}}$.

SCHOLIUMS.

1. When $A^{\frac{r}{m}}$ is rational, so also is $\overline{A^{\frac{r}{m}}}$, because this is equal to $\overline{A^{\frac{r}{m}}}$, which is the Power of a rational Root: But tho' $\overline{A^{\frac{r}{m}}}$ is rational, it does not follow that $A^{\frac{r}{m}}$ is so, as one Example shews. Thus: Let $A=3$, then $A^{\frac{4}{3}}=A^4=81$. But $A^{\frac{1}{3}}$ is Surd, for 3 has not a Cube Root.

2. Here we learn, that $A^{\frac{r}{m}}$ may indifferently be taken for the r Power of the m Root, or the m Root of the r Power of A , since these are equal. So that henceforth we shall take it either way, as shall be most useful. Hence also we have a Rule for expressing the *Involution* of a simple Root, or the *Evolution* of a simple Power.

THEOREM XI.

If the Index of a mixt Power (*i. e.* which has a fractional Index) has both its Members, Numerator and Denominator, equally multiplied or divided, the Products or Quotients put in place of the others make an equivalent Expression, or express a mixt (or simple) Power or Root of the same Number equal in Value to the former. Thus $\frac{4}{3} = \frac{8}{6}$, and therefore $A^{\frac{4}{3}} = A^{\frac{8}{6}}$. Universally, let $n=ab$, $m=ad$. Then is $A^{\frac{n}{m}}$ (or $A^{\frac{ab}{ad}}$) $= A^{\frac{b}{d}}$.

Example:

$$A = 16777216. A^{\frac{1}{6}} = 16$$

$$A^{\frac{4}{6}} = 65536 = 16^4$$

$$A^{\frac{1}{3}} = 256. A^{\frac{2}{3}} = 65536.$$

Demonstr. $A^{\frac{ab}{ad}}$ is the ad Root of the ab Power of A (by the Notation) which is the d Root of the a Root (*Theor.* 9.) of the a Power of the b Power (*Theor.* 8.) Now the b Power of A is A^b , and the a Power of this is A^{ab} , (*Theor.* 8.) whose a Root is A^b , (*Theor.* 8. Cor. 2.) and the d Root of this is $A^{\frac{b}{d}}$ (*per* Notation) *i. e.* $A^{\frac{ab}{ad}} = A^{\frac{b}{d}}$. Or the Demonstration will go on the same way by taking $A^{\frac{ab}{ad}}$ for the ab Power of the ad Root.

THEOREM XII.

THE Simple Power of a Mixt Power, or Mixt Power of a Simple, of any Number, is equal to a Mixt Power of the same Number, whose Numerator is the Product of the Simple Index by the Numerator of the Mixt one, and its Denominator that of the Mixt one; or, in short, whole Index is the Product of the two given Indexes. Thus, the r Power of $A^{\frac{n}{m}}$ is $A^{\frac{rn}{m}}$; which is also the $\frac{rn}{m}$ Power of the r Power; *i. e.* $\overline{A^{\frac{r}{m}}}$.

Example.

$$A = 4096. A^{\frac{1}{4}} = 8. A^{\frac{3}{4}} = 512$$

$$\text{Squ. of } A^{\frac{3}{4}} \text{ or } A^{\frac{6}{4}} = 32768$$

Demonstr. $A^{\frac{n}{m}}$ is the n Power of the m Root of A ; therefore the r Power of $A^{\frac{n}{m}}$ is the r Power of the n Power of the m Root: But the r Power of the

n Power is the rn Power, (*Theor.* VIII.) therefore the r Power of $A^{\frac{n}{m}}$ is the rn Power of the m Root, *i. e.* $A^{\frac{rn}{m}}$, (*per* Notation.) Again, This is also the $\frac{rn}{m}$ Power of A^r : For the Index $\frac{rn}{m}$ is the m Root of the rn Power; wherefore $\overline{A^{\frac{r}{m}}}$ is the m Root of the rn Power of the r Power; *i. e.* the m Root of the nr Power, or nr Power of the m Root; *viz.* $A^{\frac{nr}{m}}$.

THEO-

THEOREM XIII.

THE Simple Root of a Mixt Power, or Mixt Power of a Simple Root of any Number, is such a Mixt Root of the same Number, whose Denominator is the Product of the Simple Index into the Denominator of the Mixt one, and the Numerator that of the Mixt one. In short, whose Index is the Quote of the Mixt one, by the Name of the Simple one, or the Product of the two Indexes, taking the Simple Root fractionally: Thus the r Root of $A^{\frac{n}{m}}$, or the $\frac{n}{m}$ Root of $A^{\frac{1}{r}}$, is $A^{\frac{n}{mr}}$: For $\frac{n}{m} \div r = \frac{n}{m} \times \frac{1}{r} = \frac{n}{mr}$.

Example:

$A = 65536$. $A^{\frac{1}{2}} = 256$.
 $A^{\frac{3}{2}} = 16777216$. Sq. Root
of $A^{\frac{3}{2}}$, or $A^{\frac{3}{4}} = 4096$.

Demonstr. $A^{\frac{n}{m}}$ is the m Root of the n Power of A ; therefore the r Root of $A^{\frac{n}{m}}$ is the r Root of the m Root of A^n , i. e. the rm Root, (*Theor. 9.*) or $A^{\frac{n}{rm}}$. Again, this is also the $\frac{n}{m}$ Root or Power of $A^{\frac{1}{r}}$. For $\frac{n}{m}$ expresses the n Power of the m Root, and so the $\frac{n}{m}$ Root of $A^{\frac{1}{r}}$ is the n Power of the m Root of the r Root, i. e. the n Power of the mr Root, or $A^{\frac{n}{mr}}$.

THEOREM XIV.

THE mixt Power of a mixt Power is equal to a mixt Power whose Index is the Product of the given Indexes. Thus, the $\frac{r}{s}$ Power of the $\frac{n}{m}$ Power is the $\frac{rn}{sm}$ Power, i. e.

$$A^{\frac{r}{s} \times \frac{n}{m}} = A^{\frac{rn}{sm}}.$$

Demonstr. The r Power of $A^{\frac{n}{m}}$ is $A^{\frac{rn}{m}}$, (*Theor. 12.*) and the s Root of this is $A^{\frac{rn}{sm}}$ (*Theor. 13.*) which is therefore the s Root of the r Power, or the r Power of the s Root of $A^{\frac{n}{m}}$.

THEOREM XV.

THE Product of any two Roots Simple or Mixt, or of any Power and Root of the same Number, is equal to such a Power or Root of the same Number whose Index is the Sum of the given Indexes. Thus $A^{\frac{n}{m}} \times A^{\frac{r}{s}} = A^{\frac{n}{m} + \frac{r}{s}} = A^{\frac{ns + mr}{ms}}$; and $A^{\frac{n}{m}} \times A^{\frac{r}{s}} = A^{\frac{n}{m} + \frac{r}{s}} = A^{\frac{ns + mr}{ms}}$.

Demonstr. The most universal Case or Expression is $A^{\frac{n}{m}} \times A^{\frac{r}{s}}$; for by supposing n or r , or each of them, equal to 1, you make them Simple Roots; and by making m or s equal to 1, you make that Term a Simple Power: And so all the Variety supposed in the Theorem will be demonstrated in this one Form; thus: The Thing to be demonstrated

is, that $A^{\frac{n}{m}} \times A^{\frac{r}{s}} = A^{\frac{n}{m} + \frac{r}{s}} = A^{\frac{ns + mr}{ms}}$: In order to which, suppose $A^{ns} = B$, and $A^{mr} = D$; then is $B^{\frac{1}{ms}} = A^{\frac{ns}{ms}}$ (*Ax. 1.*) $= A^{\frac{n}{m}}$ (*Theor. XI.*) Also $D^{\frac{1}{ms}} = A^{\frac{mr}{ms}} = A^{\frac{r}{s}}$; and $BD = A^{ns} \times A^{mr} = A^{ns + mr}$ (*Theor. 6.*) Hence $A^{\frac{ns + mr}{ms}} = \overline{BD}^{\frac{1}{ms}} = B^{\frac{1}{ms}} \times D^{\frac{1}{ms}} = A^{\frac{n}{m}} \times A^{\frac{r}{s}}$.

THEO-

THEOREM XVI.

THE Quote of any two Roots, Simple or Mixt, or of any Power and Root of the same Number, is equal to such a Power or Root of the same Number, whose Index is the Difference of the given Indexes, when the Index of the Dividend is greater than that of the Divisor. But if the Index of the Divisor is the greater, set 1 over that Number so found, or divide 1 by it, and this Fraction or Quote, is the Quote sought. Thus $A^{\frac{n}{m}} \div A^{\frac{r}{s}} = A^{\frac{n}{m} - \frac{r}{s}} = A^{\frac{ns - mr}{ms}}$ when $\frac{n}{m}$ is greater than $\frac{r}{s}$; but it is $1 \div A^{\frac{mr - ns}{ms}}$ if $\frac{n}{m}$ is less than $\frac{r}{s}$. All other Cases are contain'd in this Form.

Demonstr. Suppose $A^{ns} = B$, and $A^{mr} = D$, then is $B^{\frac{1}{ms}} = A^{\frac{ns}{ms}} = A^{\frac{n}{m}}$, and $D^{\frac{1}{ms}} = A^{\frac{mr}{ms}} = A^{\frac{r}{s}}$; also $B \div D = A^{ns} \div A^{mr} = A^{ns - mr}$, (*Theor.* VII.) Hence $A^{\frac{ns - mr}{ms}} = \overline{B \div D}^{\frac{1}{ms}} = B^{\frac{1}{ms}} \div D^{\frac{1}{ms}} = A^{\frac{n}{m}} \div A^{\frac{r}{s}}$.

But if $\frac{n}{m}$ is less than $\frac{r}{s}$, then also is ns less than mr ; and therefore $A^{ns} \div A^{mr}$ is a proper Fraction. Let us suppose $mr = ns + a$, or $mr - ns = a$, then is $A^{ns} \div A^{mr} = A^{ns} \div A^{ns + a}$; but $A^{ns} \div A^a = A^{ns} \times A^{-a}$, (*Theor.* VI.) wherefore reduce the Fraction $A^{ns} \div A^{mr}$, or $A^{ns} \div A^{ns + a}$ to lower Terms by dividing both by A^{ns} , the new and equivalent Fraction is $1 \div A^a = 1 \div A^{mr - ns}$; but $A^{ns} \div A^{mr} = B \div D$, which is therefore $= 1 \div A^{mr - ns}$: Hence $1 \div A^{\frac{mr - ns}{ms}} = \overline{B \div D}^{\frac{1}{ms}} = B^{\frac{1}{ms}} \div D^{\frac{1}{ms}} = A^{\frac{n}{m}} \div A^{\frac{r}{s}}$.

General SCHOLIUM relating to the preceding Theorems.

FROM the preceding Theorems we have Rules for the Multiplication, Division, Involution and Evolution of Numbers expressed in Form of Powers or Roots of other Numbers, *i. e.* for a more simple and convenient Expression of the Products and Quotes, Powers and Roots; and the Substance of the whole may be represented in four General Rules, which being particularly exemplified, will shew in one short View all the preceding Theory. And observe, That in order to reduce it to so few Rules, any Expression made of any Letter A, with any Index, Integral or Fractional, may be called a *Power* of A; for it's such a Power as the Numerator expresses, of a certain Root expressed by the Denominator.

RULE I. Add the Indexes of any two Powers of the same Number A, and the Sum is the Index of a Power equal to the Product of the other two.

RULE II. The Difference of the Indexes of two Powers of A, is the Index of a Power equal to the Quote of the other two; minding, that if the Index of the Divisor is greatest, 1 is to be set over the Power found, and that fractional Expression is the true Quote.

RULE III. If the Index of a given Power of A is multiplied by another Index, the Product is the Index of a Power which is equal to such a Power of the given Power as that

that other Index denominates. And because the Word *Power* does here signify both what is in a more particular Definition call'd *Power* and *Root*, therefore this Rule comprehends both *Involution* and *Evolution*.

RULE IV. If to the Product or Quote of two Numbers any Index is applied, the Expression is equal to the Product or Quote of the same Powers of these two Numbers.

EXAMPLES of these Rules.

RULE 1.

$$\begin{array}{l} A^n \times A^r = A^{n+r} \quad \text{6th.} \\ A^n \times A^{\frac{r}{s}} = A^{\frac{n+r}{s}} \\ A^{\frac{1}{u}} \times A^{\frac{r}{s}} = A^{\frac{r+s}{us}} \\ A^n \times A^{\frac{r}{s}} = A^{\frac{ns+r}{s}} \\ A^{\frac{n}{u}} \times A^{\frac{r}{s}} = A^{\frac{ns+ur}{us}} \\ A^{\frac{1}{u}} \times A^{\frac{r}{s}} = A^{\frac{r+s}{us}} \end{array} \quad \left. \begin{array}{l} \\ \\ \\ \\ \\ \end{array} \right\} \text{15th.}$$

RULE 2.

$$\begin{array}{l} A^n \div A^m = A^{n-m}, \text{ or } A^{\frac{n-m}{1}} \quad \text{7th.} \\ A^n \div A^{\frac{r}{s}} = A^{\frac{ns-r}{s}} \\ A^{\frac{1}{u}} \div A^{\frac{r}{s}} = A^{\frac{s-r}{rs}}, \text{ or } 1 \div A^{\frac{nr}{rs}} \\ A^n \div A^{\frac{r}{s}} = A^{\frac{ns-r}{s}}, \text{ or } 1 \div A^{\frac{nr}{s}} \\ A^{\frac{n}{u}} \div A^{\frac{r}{s}} = A^{\frac{ns-ur}{us}}, \text{ or } 1 \div A^{\frac{ur-nr}{us}} \\ A^{\frac{1}{u}} \div A^{\frac{r}{s}} = A^{\frac{s-r}{us}}, \text{ or } 1 \div A^{\frac{nr-s}{us}} \end{array} \quad \left. \begin{array}{l} \\ \\ \\ \\ \\ \end{array} \right\} \text{16th.}$$

RULE 3.

$$\begin{array}{l} \overline{A^m}^n = A^{mn} \quad \text{8th.} \\ \overline{A^{\frac{1}{m}}}^n = A^{\frac{n}{m}} \quad \text{9th.} \\ \overline{A^{\frac{1}{m}}}^n \} = A^{\frac{n}{m}} \quad \text{10th.} \\ \overline{A^{\frac{n}{m}}}^r \} = A^{\frac{nr}{m}} \quad \text{12th.} \\ \overline{A^{\frac{n}{m}}}^r \} = A^{\frac{nr}{m}} \quad \text{13th.} \\ \overline{A^{\frac{n}{m}}}^r \} = A^{\frac{nr}{m}} \quad \text{14th.} \end{array}$$

RULE 4.

$$\begin{array}{l} A^n \times B^n = \overline{AB}^n \quad \text{1st.} \\ A^n \div B^n = \overline{A \div B}^n \quad \text{2d.} \\ A^{\frac{1}{u}} \times B^{\frac{1}{u}} = \overline{AB}^{\frac{1}{u}} \quad \text{3d.} \\ A^{\frac{1}{u}} \div B^{\frac{1}{u}} = \overline{A \div B}^{\frac{1}{u}} \quad \text{4th.} \\ A^{\frac{n}{u}} \times B^{\frac{n}{u}} = \overline{AB}^{\frac{n}{u}} \quad \text{5th.} \\ A^{\frac{n}{u}} \div B^{\frac{n}{u}} = \overline{A \div B}^{\frac{n}{u}} \quad \text{5th.} \end{array}$$

THEOREM XVII.

If the n Power of one Number is equal to the m Power of another, then is the m Root of the first equal to the n Root of the second: Thus if $A^n = B^m$, then is $\overline{A}^{\frac{1}{m}} = \overline{B}^{\frac{1}{n}}$.

Demonstr. Since $A^n = B^m$, then is $A^{\frac{n}{m}} = B$, (Ax. 1.) and $B^{\frac{1}{n}} = A^{\frac{n}{mn}}$, (Ax. 1.) $= A^{\frac{1}{m}}$, (Theor. XI.)

The Reverse is also true, viz. That if $\overline{A}^{\frac{1}{m}} = \overline{B}^{\frac{1}{n}}$, therefore $A^n = B^m$; for $A^{\frac{n}{m}} = B$, and therefore $A^n = B^m$.

SCHOLIUM.

SCHOLIUM. This Theorem may be made yet more universal by taking any MIXT Index, and making the Theorem thus: If $A^{\frac{n}{m}} = B^{\frac{r}{s}}$, then is $A^{ns} = B^{rm}$; and hence again, $A^{\frac{r}{s}} = B^{\frac{m}{n}}$: For the first, suppose $A^n = a$, and $B^r = b$, then is $A^{\frac{n}{m}} = a^{\frac{1}{m}}$ and $B^{\frac{r}{s}} = b^{\frac{1}{s}}$; wherefore $a^{\frac{1}{m}} = b^{\frac{1}{s}}$; and hence, (as is already shewn) $a^s = b^m$. But $a = A^n$, and $b = B^r$, therefore $a^s = A^{ns}$, and $b^m = B^{rm}$; that is, $A^{ns} = B^{rm}$; whence again, $A^{\frac{r}{s}} = B^{\frac{m}{n}}$.

COROLL. If $A^n = B^m$, and if $A^{\frac{1}{m}}$ is Rational, so also is $B^{\frac{1}{n}}$, since they are equal.

THEOREM XVIII.

IF $A^n = B^m$, call $m - n = d$. Then is A^d a Power of the Order m , and B^d a Power of the Order n ; i. e. $A^{\frac{d}{m}}$, and $B^{\frac{d}{n}}$ are both Rational, tho' they are not always equal.

Demonstr. Since $A^n = B^m = B^{n+d}$, (for $m = n + d$) multiply both by A^d ; and then $A^{n+d} = B^{n+d} \times A^d$; therefore A^d is a Power of the Order $n + d$ or m , (by Coroll. 2 Theor. II) Divide both by B^d and it is $A^n \div B^d = B^n$; therefore B^d is a Power of the Order n , (Coroll. 3. Theor. II) i. e. $A^{\frac{d}{m}}$ and $B^{\frac{d}{n}}$ are both Rational.

COROLL. If $d = 1$, that is, $m = n + 1$, then A has an $n + 1$ Root, and B an n Root, i. e. $A^{\frac{1}{m}}$ and $B^{\frac{1}{n}}$ are both rational; as they are also equal by the Theorem.

LEMMA.

If any Fraction $\frac{a}{r}$ is in its least Terms, or is not so, then, accordingly, any Power of this Fraction, as $\frac{a^n}{r^n}$, is also in its lowest Terms, or is not so. And the *Converse*, if $\frac{a^n}{r^n}$ is or is not in its lowest Terms, accordingly $\frac{a}{r}$ is or is not so.

The *Demonstration* of this Truth must be referred to another Place, because it depends upon Principles not yet explained: You'll find it demonstrated in *Book V. Ch. I. Theor. 13. Coroll.* In the mean time we must suppose it to be true, for the sake of some things belonging to this *Book*, whose *Demonstration* depends upon this *Lemma*, and which could not be so regularly referred.

THEOREM XIX.

IF any Integer A has not a proposed Root in Integers, it can have no determinate Root of that Order; i. e. it is a *Surd* Power of that Order. Or thus: If an Integer A is not the Power of a certain Order of an integral Root, it cannot be so of a fractional.

Example. 7 has no Square or Cube Root in Integers, and therefore has no such determinate Root in a Fraction.

Demonstr. Let $\frac{a}{r}$ be any Fraction in its lowest Terms, and such as is not equal to any Integer, i. e. let r be greater than 1; then, by the *Lemma*, $\frac{a^n}{r^n}$ is also a Fraction in its lowest Terms: and consequently, r^n is not an aliquot Part of A^n ; nor, consequently, is $\frac{a^n}{r^n}$ an Integer; for in this Case $\frac{a^n}{r^n}$ would not be in its lowest Terms: Hence again it's clear, that no

no Fraction (such as is not equal to an Integer) can be any Root to a Whole Number: For suppose $\frac{s}{m}$ to be the n Root of A , then is $\frac{s^n}{m^n} = A$; but if $\frac{s}{m}$ is in its least Terms, so is $\frac{s^n}{m^n}$; therefore $\frac{s^n}{m^n}$ not an Integer is equal to A an Integer, which is absurd. Again, if $\frac{s}{m}$ is not in its least Terms, let $\frac{a}{r}$ be its least Terms; then, because $\frac{s}{m} = \frac{a}{r}$ therefore $\frac{s^n}{m^n} = \frac{a^n}{r^n} = A$. But $\frac{a}{r}$ being in its least Terms, so is $\frac{a^n}{r^n}$; whence the same Absurdity as before.

CHAP. II.

Containing the PRACTICE of Involution and Evolution.

§. I. Probl. Of INVOLUTION, or Raising of Powers.

THE General Rule for the Practice of *Involution* is plainly contained in the Definition, and is nothing else but a continued Multiplication of the Root into it self, whereby, to come at any higher Power, we must make up the Series of all the interior Powers: Thus; the Series of the Powers of 3 is 3:9:27:81:243, &c. By this Operation $3 \times 3 \times 3 \times 3 \times 3$, &c.

But there is a particular Method whereby all Powers above the Cube or 3d Power may be found, without actually finding all the inferior Powers: Which Rule is this:

RULE. Find, by the general Rule, two or more such Powers of the given Root as that the Sum of their Indexes be equal to the Index of the Power required; then multiply these Powers continually into one another; the Product is the Power sought. Or find any one Power whose Index is an aliquot Part of the Index of the Power sought, and involve that Power to an Index equal to the Denominator of that aliquot Part.

Example 1. To find the 7th Power of 4, I find the 2d, 3d, and 4th Powers; viz. 16, 64, 256; then the 3d \times 4th, or $64 \times 256 = 16384$ the 7th Power; because $3 + 4 = 7$. Or instead of finding the 4th Power I find the 5th, by multiplying the 2d and 3d; viz. $16 \times 64 = 1024$; then the 2d \times 5th, or $16 \times 1024 = 16384$ the 7th Power.

Example 2. To find the 12th Power of 3, I find the Square, viz. $3 \times 3 = 9$; the Square of the Square, or $9 \times 9 = 81$ is the 4th Power; again, the 2d \times 4th, or $9 \times 81 = 729$ the 6th Power; then 6th \times 6th, or $729 \times 729 = 531441$ the 12th Power.

I need insist no more on this Practice; the Reason of which is plainly contained in the preceding Theor. VI. But I must observe, That it does not in every Case give any Advantage either of Ease or Expedition to the Work; yet as it will do so in many Cases, and in none can it make the Operation more tedious, it will always be a very convenient Method.

Of the Practice in Universal Characters.

As to the literal Practice, or Involution of Numbers represented by Letters and Indexes, this is also sufficiently explained in the preceding Definitions and Theorems.

X

But

But observe, That when a Root is represented as a complex Quantity; for Example, If instead of S we put $A+B$, its Powers may be represented two different Ways; Thus, $\overline{A+B}^2$ $\overline{A+B}^3$ or $\overline{A+B}^n$; which Method is in some Cases sufficient; but in others it's necessary to have the Operation performed, and the Power expressed according to the Result of the Multiplication; so that the Index be applied only to the single Letters: Thus, $\overline{A+B}^2 = A^2 + 2AB + B^2$. The most considerable and important of all these Cases of complex Roots, with their Powers, is that wherein there are only two Members in the Root, as $A+B$, called hence a *Binomial* Root; or $A-B$, called a *Residual* Root; the Consideration of whose Powers, *i.e.* of their Composition by the various Powers and Multiples of the different Members of the Root, has been found of very great Use in Mathematicks, and in Arithmetick, especially for the Business of the Extraction of Roots; in order to which I shall here explain it.

Of the Composition of the Powers of a Binomial and Residual Root.

In the annex'd Operation you see the several Powers of $A+B$ raised by Multiplying, according to the common Rules, each Member of the Root into each Member of the several Powers, which produces the next Powers; in which these Things are remarkable.

Observations on the Table of Powers.

I. In the Expression of each Power there are as many and no more different Members, (which contain different Powers of the Parts of the Root A and B) as the Index $+1$ expresses: For tho' each Member

TABLE of Powers raised from the Root $A+B$.

Root.	$A+B$
	$A+B$
	$A^2 + AB$
	$+ AB + B^2$
Square.	$A^2 + 2AB + B^2$
	$A+B$
	$A^3 + 2A^2B + AB^2$
	$+ A^2B + 2AB^2 + B^3$
Cube.	$A^3 + 3A^2B + 3AB^2 + B^3$
	$A+B$
	$A^4 + 3A^3B + 3A^2B^2 + AB^3$
	$+ A^3B + 3A^2B^2 + 3AB^3 + B^4$
4th Power.	$A^4 + 4A^3B + 6A^2B^2 + 4AB^3 + B^4$
	&c.

of any of the Powers being multiplied by A and B separately, do make in all twice as many Products as the Terms in the Power multiplied; yet each of the Series of Products by A and B have all their Terms similar, except the first Product by A , and the last by B ; for these two are A^n and B^n ; *i.e.* the two Series of Products by A and B contain in their several Members the same Powers of A and B , except the first Term of the Line of Products by A , which is A^n , and the last Term of the Line by B , which is B^n ; consequently, these similar Products are reducible to a more simple Expression by adding them together, (*i.e.* adding the Numbers by which they are multiplied, and joining the common or similar Letters with their Indexes) thus, $A^3B + 3A^3B = 4A^3B$; also $3A^2B^2 + 3A^2B^2 = 6A^2B^2$; which explains the Reason of placing the Lines of Products by A and B , as is here done, *viz.* in order to the Addition of similar Products. That this Observation will hold true however far the Powers are carried, is easily seen from the Nature of the Thing; which will yet farther appear from the next Observation, in which we have a joint Demonstration of this.

II. Each Power of $A+B$ contains a Series of gradually different Powers of A and of B : Thus, The Index of any Power of $A+B$ being n , the first Term is simply A^n , and the last

last is B^n ; the intermediate Terms containing each a different Power of both A and B, multiplied together; the Index of A decreasing gradually by 1 in each Term from A^n to the Term preceding the last, or B^n , in which it is simply A; and the Indexes of B increasing the same way from the Term next after A^n , in which it is only B, to the last, or B^n ; so that in all the intermediate Terms there is some Power of A and of B; and the Sum of their Indexes is equal to n , the Index of the Power proposed of $A+B$. Therefore, omitting the other Numbers, which are Multipliers in the several Terms of any Power of a Binomial, whose Index is n , these Terms, in as far as they are composed of the Powers of A and B, may be represented thus;

$$A^n + A^{n-1} \times B + A^{n-2} \times B^2 + A^{n-3} \times B^3 + \&c. A^2 \times B^{n-2} + A \times B^{n-1} + B^n.$$

wherein there are as many Members as $n+1$, according to the first Observation; and the Index of B, or the Number taken from n in the Index of A, is the Number of Terms after A^n to any Term.

This Observation we see to be true so far as the Table of Powers is carried; and that it must be true for ever, is easy to perceive. Or it may be demonstrated, *thus*: Suppose it's true in any one Case or Power of $A+B$, as the n Power, it must be true in the next Case, or the $n+1$ Power: because when each Term of the given Power is multiplied by

$$A^n + A^{n-1} \times B + A^{n-2} \times B^2 + A^{n-3} \times B^3 + \&c. + A \times B^{n-1} + B^n$$

$$A+B$$

$$A^{n+1} + A^n \times B + A^{n-1} \times B^2 + A^{n-2} \times B^3 + \&c. + A^2 \times B^{n-1} + A \times B^n$$

$$+ A^n \times B + A^{n-1} \times B^2 + A^{n-2} \times B^3 + \&c. + A^2 \times B^{n-1} + A \times B^n + B^{n+1}$$

A, the Products must have A once more involved in them than in the Term multiplied: And since the Indexes of A decrease gradually from A^n

in the Terms multiplied, consequently they will decrease gradually from A^{n+1} in the Series of Products; the Powers of B continuing as they were. Again; The Series of Products made by B must have B once more involved in each; and consequently increasing gradually from B to B^{n+1} , leaving the Powers of A as they were: But again, These Products made by B are all of them, (except the last B^{n+1}) similar to the several Products, (after the first A^{n+1}) made by A; because the Indexes of A in the given Power decreasing from the first Term A^n , which has no Power of B multiplied into it, and those of B increasing from the second Term $A^{n-1} \times B$ to the last Term B^n , it's plain that A multiplied into any Term except the first A^n , and B multiplied into the preceding, must make similar Products; for A multiplied into any Term raises the Index of the Power of A by 1, which makes it equal to the Index of A in the preceding Term, without changing that of B; and B multiplied in the preceding Term, raises the Power of B in it to the Index of B in the following Term, without changing that of A; consequently these Products; are similar, which makes the thing observed manifestly true in any Case, in consequence of its being true in the preceding: But it's true in the Root or 1st Power, and as far as we have carried the Powers, therefore it's universally true. And this also is manifest, that the Sum of the Indexes of A and B that are in any Term, is always equal to the Index of the Binomial Power, *viz.* n . Add also this Observation, that the Index of A or B is always 1 less than the Number of Terms from A^n or B^n , to that Term.

SCHOLIUM. If any one Member of a Binomial is 1, as $A+1$, then the Powers of 1 being all 1, the Powers of such a Root will consist only of the Series of the Powers of A, and 1 added; thus, $A^n + A^{n-1} + A^{n-2} + \&c. + 1$. Or thus, $1 + A + A^2 + \&c. + A^n$.

III. The Numbers which in every Power are multiplied into the several Terms are called *Coefficients* (*i.e.* joint Multipliers or Factors) of these Terms; and from the Manner of raising the

the Powers this is to be observed, That the Coefficients of the first and last Terms are 1, and those of the intermediate Terms are each the Sum of the Coefficients of the corresponding and preceding Terms of the preceding Power; thus, The Coefficient of the third Term of the 4th Power is 6, equal to $3 + 3$, the Coefficients of the 3d and 2d Terms of the 3d Power, (see the preceding *Table of Powers*.) Now that you may perceive the Reason of this, and that it must continue so for ever in all Powers, consider these two Articles:

(1.) The Products of the several Terms of any Power, made by A or by B, do not change the Coefficients of the Terms multiplied, because A and B have no Coefficient but 1. Then

(2.) The similar Products made by A and by B are these, *viz.* The Product of the 2d Term, (of the Power multiplied) by A, and the Product of the 1st Term by B; the 3d Term by A, and the 2d Term by B; and so on. Which similar Products are added by the adding of their Coefficients, and annexing the similar Parts or Powers of A and B.

Now from these two Things the universal Truth of the Observation is manifest; and the annex'd Table, so far as it is carried on by this Rule, shews the Series of Coefficients of any Power of a Binomial.

TABLE of Coefficients of the Powers of a Binomial Root.

	Coefficients.
1st	1 : 1
2d	1 : 2 : 1
3d	1 : 3 : 3 : 1
4th	1 : 4 : 6 : 4 : 1
5th	1 : 5 : 10 : 10 : 5 : 1
6th	1 : 6 : 15 : 20 : 15 : 6 : 1
7th	1 : 7 : 21 : 35 : 35 : 21 : 7 : 1
8th	1 : 8 : 28 : 56 : 70 : 56 : 28 : 8 : 1
9th	1 : 9 : 36 : 84 : 126 : 126 : 84 : 36 : 9 : 1.
	<i>&c. &c.</i>

COROLL. From the two last Observations we learn a new and easier Way than the common, for raising any Power of a Binomial Root. Thus: take the Series of Products of the Powers of A and B, according to the second Observation; and to these apply the proper Coefficients, as they stand in this Table; and if you have no such Table, you must raise one, as far as the proposed Power; which being done by simple Addition, is much easier than the common Rule. Thus, for Example; The 4th Power of $A + B$ is

$$A^4 + 4 A^3 B + 6 A^2 B^2 + 4 A B^3 + B^4.$$

But, again, to make this yet easier, see the following Observation, and its Corollary.

IV. Any two Coefficients in the Series belonging to each Power are the same Numbers, if they are taken at equal Distances from the Extremes, (which have both 1) for the Coefficients increase from the one Extreme to the Middle Term, where there is one Middle Term, and decrease from that to the other Extreme by the same Series by which they increased; and if there are two Middle Terms, these are equal, and they decrease upon each hand by the same Numbers to the Extremes. The universal Truth of this is manifest from the way of constructing the Table: For being true in any Case, (as we see it is as far as the Table is carried) it must be true in the next Case or Power, and so on for ever. And hence again, if we call the Place of any Term from the one Extreme a , the other Term whose Coefficient is equal, is from the same Extreme in the Place expressed by $n - a + 2$ (n being the Index of the Power) For the whole Number of Terms is $n + 1$, by *Observ. I.* And that Term which is in the a Place, from the one Extreme, must necessarily be in the $n + 1 - a + 1 = n - a + 2$ Place from the other Extreme. And since Coefficients at equal Distances from the two Extremes are equal; hence it is, that reckoning them both from the same Extreme, their Places are a and $n - a + 2$. Again, If we call the Index $n = a + b - 2$. (*i.e.* add 2 to the Index, and suppose the Sum $n + 2 = a + b$, whereby $n = a + b - 2$) then are the Coefficients equal which are in the a and b Places from

from the same Extreme; for $b = n - a + 2$; and we have seen already that the Coefficients, in the a and $n - a + 2$ Places are equal.

In the last Place, take Notice, That the 2d Term from either Extreme has for its Coefficient the Index of the Power.

COROLL. Hence, in making up the Table of Coefficients for any Power, as $A + B^n$, when we are come to that Series which has as many Terms as $\frac{n+1}{2}$ i. e. the half Number of Terms belonging to the proposed Power n , when that Number $n+1$ is an even Number; or that has as many Terms as $\frac{n+2}{2}$ when $n+1$ is an odd Number; we need raise the following Series of the Table to no more Terms, till we come to the proposed Power n ; and the remaining Terms of it are the same with these already found, taken in a reverse Order, as above explained. Thus: To find the Coefficients of the 9th Power, which has ten Terms; when you have arrived, in making the Table, at the 4th Power, which has five Terms, you need raise no more Terms in the following Series till you come to the 10th, and then make the remaining five Terms of it the same with the preceding, in a reverse Order. And for the 8th Power, which has nine Terms, you must also have the Coefficients compleat to the 4th Power, which has five Terms; and when you come to the 8th, the remaining four are the same with the first four reversely.

V. The preceding Observations were all obvious: But the following most valuable Property of the Coefficients, in which we have a curious Rule for finding the Coefficients of any Power without regard to the preceding Powers, we owe to the happy Genius of the incomparable Sir ISAAC NEWTON; which is this, viz.

Rule. The Coefficient of any Term is equal to the Product of the Coefficient of the preceding Term multiplied into the Index of A in that preceding Term, and divided by the Number of Terms from A^n to that Term: And because the Coefficients of the first and second Terms are always known, which are 1 and n , by *Observ. 4.* therefore it is plain, that by this Rule the Series of Coefficients of any Power may be found independently of preceding Powers.

Exam. The Coefficient of the fourth Term of the eighth Power is 56, the Index of A in that Term is 5; then $56 \times 5 = 280$, and $280 \div 4 = 70$, which is the 5th Term.

In order to the Demonstration of this Rule, we shall first explain the universal Expression of it in Letters, which is this: Take the Index of the Power n , and make this Series of Factors, $1 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \dots$ carrying it to a Number of Terms equal to $n+1$; and the first Term or 1 is the Coefficient of the first Term of the Power; $1 \times \frac{n}{1}$ or simply n is the second Coefficient; $1 \times \frac{n}{1} \times \frac{n-1}{2}$ is the third Coefficient, and so on, taking in always one Factor more at every Step, till you have all the Coefficients belonging to that Power, which are in Number $n+1$. But, as is before observ'd, having found them for the one half of the Terms, or to the middle Term, the other half is found also without any farther Operation.

Now that this is a true and just Expression of the preceding Rule, will be plain from these Considerations: 1. That the first and second Terms are in all Cases 1 and n . 2. That the Index of A decreases continually from A^n the first Term; A^{n-1} being the second, and so on; whereby it is manifest, that according to this Rule with that in *Observ. 2.* the n Power of $A + B$ is represented as in the following Series; which is called

The Binomial Theorem.

$$\begin{aligned} A+B^n = & 1 \times A^n + 1 \times n \times A^{n-1} \times B + 1 \times n \times \frac{n-1}{2} \times A^{n-2} \times B^2 + 1 \times n \times \frac{n-1}{2} \times \frac{n-2}{3} \times A^{n-3} \times B^3 \\ & + 1 \times n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times A^{n-4} \times B^4 + \dots \text{ \&c. which is carried to a Number of Terms} \\ & \text{equal} \end{aligned}$$

equal to $n+1$; and then the last Term will be B^n ; the Index of A being 0, whereby A is out of that Term; and the Coefficient is 1.

Before we come to the *Demonstration*, we must observe upon this Expression of the Rule for the Coefficients, that the Numbers taken from n in the Numerators are always 1 less than the Denominators; and these (which are also equal to the Index of B, or the Number taken from n in the Index of A) being in Arithmetical Progression increasing from 1, the Numerators are in Arithmetical Progression decreasing from n . Hence the Denominator of the last Factor in each Coefficient is the Number of Factors after 1; and is also the Number of Terms after A^n to that Term; wherefore if the Denominator of the last Factor of any Coefficient is called a , that Term is in the $a+1$ place from the beginning; or if it is in the a Place, that Denominator is $a-1$. Wherefore the Coefficient of the a Place of the n Power is $1 \times n \times \frac{n-1}{2}$, &c. carried on till the last Factor, is $\frac{n-a+2}{a-1}$; or, make a the Place of the Term after the first, i. e. the Number of Terms -1 ; then the Coefficient is $1 \times n \times \frac{n-1}{2}$, &c. to $\frac{n-a+1}{a}$. And if we take this Series of Factors backwards, it is $\frac{n-a+2}{a-1} \times \frac{n-a+3}{a-2}$, &c. to 1, when a is the Number of Terms; or $\frac{n-a+1}{a} \times \frac{n-a+2}{a-1}$, &c. to 1, when $a+1$ is the Number of Terms. We shall next demonstrate the universal Truth of this Rule for Coefficients. Thus:

Demonstration of the preceding Rule for COEFFICIENTS.

1. If the Rule is good in any one Case or Power of $A+B$, as the n Power, it must therefore be good in the next Power, or $n+1$. To prove this Connection, see the two following Series; whereof the first contains the Coefficients for the n Power, according to the Rule; and the other the Coefficients for the $n+1$ Power, according to the same Rule; and because 1 does not multiply, I have omitted it in all the Coefficients but the first, which is itself 1.

For the n Power, 1, $\frac{n}{1}$, $\frac{n}{1} \times \frac{n-1}{2}$, $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$, $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, &c.

For the $n+1$ Power, 1, $\frac{n+1}{1}$, $\frac{n+1}{1} \times \frac{n}{2}$, $\frac{n+1}{1} \times \frac{n}{2} \times \frac{n-1}{3}$, $\frac{n+1}{1} \times \frac{n}{2} \times \frac{n-1}{3} \times \frac{n-2}{4}$, &c.

By *Observ.* 3. the Coefficients of any Power of $A+B$ are each equal to the Sum of the Coefficients of the corresponding and preceding Terms of the preceding Power of $A+B$; wherefore the first Series being the true Coefficients of the n Powers, the second will be the true Coefficients of the $n+1$ Power; providing that they have this Connection with the former, viz. that any Term is the Sum of the corresponding and preceding Terms of that former; which is therefore the thing to be proved, Thus:

The first Coefficient in all Powers is 1; then the Sum of the first and second Coefficients of the n Power is $1 + \frac{n}{1} = \frac{n+1}{1}$ the second Term of the second Series. Again, the

Sum of the second and third Terms of the first Series is $\frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} = \frac{n}{1} \times 1 + \frac{n-1}{2}$
 $= \frac{n}{1} \times \frac{n+1}{2} = \frac{n+1}{1} \times \frac{n}{2}$ (by changing the Order of the Numerators, which does not change the Product) and this is the third Term of the second Series; then the Sum of the

the third and fourth Terms of the first Series is $\frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} = \frac{n}{1} \times \frac{n-1}{2}$

$\times 1 + \frac{n-2}{3} = \frac{n}{1} \times \frac{n-1}{2} \times \frac{n+1}{3} = \frac{n+1}{1} \times \frac{n}{2} \times \frac{n-1}{3}$ (by changing only the Order of

the Numerators, which does not change the Product) and this is the fourth Term of the second Series. From the Nature of these Series, it's evident they must have, every-where, the same Connection; or we may also shew the Universality of it. Thus:

By what's shewn in the Observation made upon the Expression of this Rule, the Coefficient of any Term, as that in the a Place after the first or A^n , may be thus expressed,

$1 \times \frac{n}{1} \times \frac{n-1}{2} \times \dots \times \frac{n-a+2}{a-1} \times \frac{n-a+1}{a}$, and the preceding Term will be $1 \times \frac{n}{1} \times \frac{n-1}{2} \times$

$\dots \times \frac{n-a+2}{a-1}$, which contains all the Factors of the other except the last. The Sum

of these two is therefore, $1 \times \frac{n}{1} \times \frac{n-1}{2} \times \dots \times \frac{n-a+2}{a-1} \times 1 + \frac{n-a+1}{a} = 1 \times \frac{n}{1} \times$

$\frac{n-1}{2} \times \dots \times \frac{n-a+2}{a-1} \times \frac{n+1}{a} = 1 \times \frac{n+1}{1} \times \frac{n}{2} \times \frac{n-1}{3} \times \dots \times \frac{n-a+2}{a}$, (by chan-

ging the Order of the Numerators) which is the a Term after the first in the $n+1$ Power; because the Denominators are the same Series, 1, 2, 3, &c. to a , which are the Denominators in all Powers; and the Numerators decrease gradually from the Index $n+1$; so that the Number subtracted from the Index $n+1$ in the last, is less by one than the Denominator, (as has been observed and explained upon this Expression of the Rule); for the last Numerator is here $n-a+2 = n+1-a+1 = n+1-a-1$.

2. But this Rule is true when applied to the first Power, and to all the Powers as far as we have raised them in the preceding Table; therefore, by what's now shewn, it's true in the next Power above; and consequently in all above, *i. e.* in all the Powers whatever of $A+B$.

SCHOLIUMS.

1st. Different Authors have made different Demonstrations of this Rule; I have chosen what I think as easy as any of them, and fittest for this place. In *Book V. Chap. 6.* you'll find another Demonstration of it from Principles which have not, as I know, been applied to this purpose.

2. If instead of a Binomial $A+B$ we take a Residual $A-B$, it's manifest that all the Difference betwixt its Powers and these of $A+B$ will be, That whereas all the Members of the Binomial Powers are added together, these of the Residual Powers will be connected with the Signs of Addition and Subtraction, alternately; but the Powers of A and B , with the Coefficients are the very same: Thus, $\overline{A-B}^2 = A^2 - 2AB + B^2$, and $\overline{A-B}^3 = A^3 - 3A^2B + 3AB^2 - B^3$; also $\overline{A-B}^4 = A^4 - 4A^3B + 6A^2B^2 - 4AB^3 + B^4$, and so on.

3. In applying this Rule for finding any Coefficient, (either of a Binomial or Residual) observe to take its Place from the nearest Extreme, A^n or B^n , which will make the Operation shorter, and produce the same Number, since the Coefficients are the same Series of Numbers, from either Extreme. Thus, to find the Coefficients in the a Place (from either Extreme) of the n Power: Compare a , and $n-a+2$ (for, by *Observ. 4.* the Coefficients in the a , and $n-a+2$ Places, from the same or different Extreme, are equal.) Which ever of these Numbers is least, find the Coefficient for that Place. *Example:* To find the 7th Coefficient of the 10th Power; I find the 5th Coefficient, which is equal to the 7th; for if $n=10$. $7=a$, then is $n-a+2=5$. 4. Tho'

4. Tho' we had taken no notice of the Equality of Coefficients at equal Distances from the two Extremes, yet the Rule now demonstrated would have shewn it of itself: Thus. The Theorem for Coefficients is $1 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \dots \times \frac{n-a+1}{a}$; which is the Coefficient in the $a+1$ Place, or the \bar{a} Place after the first. Now the Numerators decrease from n to $n-a+1$, or $n-\bar{a}+1$, by a constant Difference 1, and the Denominators increase from 1 to \bar{a} . Again; Since the Number of Terms in the n Power is $n+1$, and in every Coefficient there are as many Factors as the Number of Terms from the Beginning, or A^n ; therefore, if we want the last Coefficient, or that in the $n+1$ Place, then is $a=n$; and consequently, $a+1=n+1$, and $n-a+1=n-n+1=1$; so that the Numerators and Denominators are the very same Series of Numbers, only in different Order, which alters not the Product; and being equal, therefore the Product is $=1$. Let us now express the Series thus; $1 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \dots \times \frac{3}{n-2} \times \frac{2}{n-1} \times \frac{1}{n}$; it's plain the last Coefficient but one, is the Product of this Series, excluding $\frac{1}{n}$; which Product is equal to $1 \times \frac{n}{1}$, for all the other Factors upon each hand of the middle one, (whose Numerator and Denominator must be equal) are reciprocal to one another, and so make the Product of them all only 1; or if there are two middle ones they are Reciprocals. By the same Reason, the Coefficient in the last Place but two, is $1 \times \frac{n}{1} \times \frac{n-1}{2}$: For excluding $\frac{2}{n-1} \times \frac{1}{n}$, the middle Terms after $\frac{n-1}{2}$ destroy one another's Effect, and make their total Product no more than 1: The same Reasoning holds in every Place. And hence again observe, that if we apply the Rule to find a Coefficient standing from the first Place further than the middle Place, or the last of two middle Places; then whenever in writing down the Factors, we come to one whose Numerator and Denominator are equal, or to two adjacent Factors that are Reciprocals, there we may stop; for what follows will destroy the Effect of as many of these preceding that one whose Numerator and Denominator are equal, or the first of these two adjacent Reciprocals, as the remaining Number to be yet set down; and therefore, by cutting off so many of the Factors (as leaves a Number equal to the Place of the Coefficient sought, number'd from the nearest Extreme) we have what's sought: Thus; For the 8th Coefficient of the 10th Power, it is $1 \times \frac{10}{1} \times \frac{9}{2} \times \frac{8}{3} \times \frac{7}{4} \times \frac{6}{5} \times \frac{5}{6} \times \frac{4}{7}$; which is $= 1 \times \frac{10}{1} \times \frac{9}{2} \times \frac{8}{3}$, for the rest destroy one another, being Reciprocals.

5. If we take the perpendicular Columns of the Table of Coefficients, it's plain these are Coefficients all in the same Place, or Distance from the Beginning in different Powers; and may be called *Similar Coefficients* of different Powers. Again; We have explained above, that if the Place of any Coefficient is a , the last Factor that composes it is $\frac{n-a+1}{a-1}$, and so that Coefficient will be $1 \times \frac{n}{1} \times \frac{n-1}{2} \times \dots \times \frac{n-a+1}{a-1}$; then by changing the Value of n this will express all the similar Coefficients in the a Place of different Powers: observing this, That the lowest Value we can put upon n is $a-1$; because no Power below that of the Order $a-1$ can have a Number of Terms equal to a (by *Observ. 1.*); and if $n=a-1$, the Coefficient will be 1; for it is the a Coefficient of the $a-1$ Power, which being the last Coefficient, is therefore 1, and is consequently the first Term of the Series of similar Coefficients of different Powers, from that whose Index is $a-1$: So that by taking n successively equal to $a-1$, a , $a+1$, &c. we shall have the Series of Coefficients of the \bar{a} Place of those different Powers whose Indexes are $a-1$, a , $a+1$, $a+2$, &c. But

But we may express this Rule also thus: Instead of x put $a + b - 2$, and it is $1 \times \frac{a+b-2}{1} \times \frac{a+b-3}{2} \times \text{&c.}$ to $\frac{a+b-2-a-2}{a-1}$ or $\frac{b}{a-1}$; which, according to the general Rule of Coefficients, is the Coefficient of the a Term of the $a + b - 2$ Power; and by taking b successively equal to 1, 2, 3, &c. we shall have hereby the Series of Coefficients in the a Place of all Powers from the $a - 1$ Power; for if $b = 1$, then is $a + b - 2 = a - 1$, and the Rule gives the first similar Coefficient, which is always 1; if $b = 2$, then $a + b - 2 = a$, and we have the second similar Coefficient; if $b = 3$, then $a + b - 2 = a + 1$, and we have the third similar Coefficient, and so on: Or if we take $m = a + b$, the Rule is $1 \times \frac{m-2}{1} \times \frac{m-3}{2} \times \text{&c.} \times \frac{m-a}{a-1}$.

Hence again we have this general Truth to observe, *viz.* That the a Coefficient of any Power whose Index is $a + b - 2$, is the same as the b Term of the Series of similar Coefficients which are in the a Place of different Powers. And this will easily be proved from these two Considerations: The 1st is what we have already explained, *viz.* That if the Index of any Power is $a + b - 2$, then the a Coefficient of that Power is equal to its b Coefficient (See *Observ. 4*) The 2d is, That from any Term in the Table of Coefficients, (*i. e.* any Coefficient of any Power) there stand as many Terms on the right hand, as there are Terms above it in the perpendicular Column of similar Coefficients; and therefore that Term is in the same Place of the similar Coefficients, and of the Line of Coefficients of that Power, numbering from the right hand: Wherefore it's plain, that the a Coefficient (reckoning from the left hand) of the $a + b - 2$ Power, is the same as the b Term of the Column of similar Coefficients in the a Place of different Powers, because it's the same as the b Coefficient from the Right of the same Power.

COROLLARIES.

1. These Expressions of Powers of a *Binomial* Root shew us how the Difference betwixt any two similar Powers is composed of the various Powers and Multiples of any one of the Roots, and the Difference betwixt the Roots: Thus, A being one Root, and B the Difference of that and another $A + B$, or $A - B$, the Difference of their Squares is $2 AB + B^2$. Hence

Having any Power of any Root, we can find another similar Power whose Root shall differ from the given one by any Difference, and that without either knowing or enquiring what that other Root is.

For *Example*: 144 is the Square of 12; and if the Difference betwixt this Root and another is 9, hence the Square of that other is $144 + 2 \times 12 \times 9 + 81 = 144 + 216 + 81 = 441$, if 12 is the lesser Root; but it is $144 - 216 + 81 = 9$, if 12 is the greater Root.

Observe, If the given Difference B is 1, then in all the Terms wherein there is any Power of B , we have nothing but the Powers of A , with the Coefficients; except the last Term B^n , which stands alone, and is 1; for $A + 1^2 = A^2 + 2A + 1$, and $A + 1^3 = A^3 + 3A^2 + 3A + 1$.

2. We have here also learnt another Way of Raising a given Number to any Power, by means of the similar Powers of the Binomial: Thus;

Take all the significant Figures of the given Number in their compleat Value, as so many different Members that compose it, by Addition; then take the two highest, calling them A and B : Raise this Binomial to the proposed Power; then consider the first two Members as one; call their Sum again A , and call the next Member B , and raise this new Binomial to the same Power; in doing of which, *observe*, that so much of the Work is already done, because the n Power of the first Member of the present Binomial is the total Power of the preceding Binomial, which is already found; so that what remains is to

make up the other Members of the Power sought, according to the general Canon. In the same manner consider the three highest Members as one, and call it A, and join the next Member, calling it B, and raise this Binomial; and thus proceed till all the Members are taken in.

Example: To find the Square of $246 = 200 + 40 + 6$, the Operation is

$$\begin{aligned} \overline{200 + 40}^2 &= 40000 + 16000 + 1600 = 57600 \\ \overline{A + B}^2 &= A^2 + 2AB + B^2 \end{aligned}$$

$$\begin{aligned} \text{Then } \overline{240 + 6}^2 &= 57600 + 2880 + 36 = 60516 \text{ the Square sought.} \\ \overline{A + B}^2 &= A^2 + 2AB + B^2 \end{aligned}$$

If there are more Members, you must go on the same Way.

Example 2. To find the Cube of $235 = 200 + 30 + 5$, the Work is

$$\begin{aligned} \overline{200 + 30}^3 &= 8000000 + 3600000 + 540000 + 27000 = 12167000 \\ \overline{A + B}^3 &= A^3 + 3A^2B + 3AB^2 + B^3 \end{aligned}$$

$$\begin{aligned} \text{Then } \overline{230 + 5}^3 &= 12167000 + 793500 + 17250 + 125 = 12977875 \text{ the Cube sought.} \\ \overline{A + B}^3 &= A^3 + 3A^2B + 3AB^2 + B^3 \end{aligned}$$

SCHOLIUM. As to this Method of raising Powers, it's more tedious than the common Way, and therefore not to be recommended for Practice; the Design of considering it here being only for the sake of a particular Illustration to be made by it of the Rules of Extraction.

§. II. Of EVOLUTION, or Extraction of Roots.

PART I. Of Whole Numbers.

Problem I. To extract the Square Root of a Whole Number.

RULE I **I**N Order to the Solution of this Problem, we must have a Table of simple Squares, or Squares of Numbers from 1 to 9, as here in the Margin: Then

II. Beginning at the Right hand, distinguish the Figures of the given Number into Periods of two Figures as long as you can, by putting a Point over the first Figure, and over every other Figure, *i. e.* passing one, take the next.

Example: 1849 is pointed thus, 18̇49, and 34968 thus, 3̇4968; the pointed Figure being the first of each Period, and that on its Left the other; tho' the last Period may sometimes have but one Figure.

The given Number being thus pointed, the Number of Points or Periods shews us how many Figures the Root consists of; to find which we proceed thus:

III. Take the last Period, (or that next the Left) and seek it, or the next lesser Number you can find, in the Table of simple Squares, the Root of this is the first Figure on the Left of the Root sought; which being written down to the Right of the given Number, as we do the Quote in Division, then set down its Square under the last Period, and take their Difference, to which prefix the next Period of the given Number: And all this taken for one Number

Roots.	Squares.
1, 1	
2, 4	
3, 9	
4, 16	
5, 25	
6, 36	
7, 49	
8, 64	
9, 81	

ber, as it stands, we call the Second Resolvend. because out of it we seek the next Figure of the Root, (the last Period being the first Resolvend) thus:

IV. Consider the Figure found as having 0 before it, and then multiply it by 2, (which is in effect, multiplying it by 20); make this Product a Divisor, and find how oft it is contained in the Resolvend; which, to the present purpose, must not be taken above nine times; tho' in some Cases it may be oftner contained; and then also it must be under this Limitation, viz. that the Square of the Quote, or Figure now set in the Root, added to its Product by the Divisor, the Sum do not exceed the Resolvend: Or, which is the same thing, put the Quote, or suppose it put, in the Place of the 0 which stands in the Place of Units of the Divisor; then multiplying the whole by the Quote, the Product must not exceed the Resolvend: For if it do, the Figure taken is too great, and you must try a lesser, till it answer. The Figure thus found is the next Figure of the Root sought, which must be set on the Right of the last: And setting the Sum or Product mention'd under the Resolvend, take their Difference, to which prefix the next Period of the given Number; and all this considered as one Number, is your next Resolvend; out of which the next Figure is to be sought thus:

V. Take both the Figures of the Root found, as they stand, for one Number; double it, and prefix 0, (or prefix 0, and then double, which is the same thing,) and this is your Divisor: Find how oft it is contained in the Resolvend, under the same Limitations as formerly; place the Figure found on the Right of these before found, and subtracting the Product directed to be compared with the Resolvend from it; to the Difference prefix the next Period, and you have the next Resolvend; to which make a Divisor out of the Figures of the Root already found, the same way as before; and thus proceed till all the Periods are employed, finding a new Figure of the Root for every Period: And if at any Step the Divisor is greater than the Resolvend, or if 1 added to the Divisor makes the Sum greater than the Resolvend; then place 0 in the Root, and prefix another Period, forming a new Divisor by setting another 0 to the former Divisor, and so go on.

E X A M P L E S.

Ex. 1. To find the Square Root of 1369, it is 37, as found by this

Operation.

$$\begin{array}{r}
 1369 \quad \left(\begin{array}{l} a. \ b \\ 3 \ 7 \end{array} \right. \\
 \underline{a^2 = 9} \\
 2a = 60 \left. \begin{array}{l} 469. \text{ 2d Resolvend.} \\ 469 = 2ab + b^2 \end{array} \right) \\
 \underline{(a=30) } \\
 000 \\
 \hline
 \end{array}$$

Explication.

The given Number being pointed, the last Period is 13, and the next Square to this is 9, whose Root is 3, which is the last Figure of the Root; and calling it a , I take $a^2 = 9$ out of 13, and to the remaining 4 I prefix the next Period 69, which makes 469 the 2d Resolvend: Then taking $a = 30$, I double it, and make $2a = 60$ a Divisor; and seeking how oft it is contained in 469, under the Limita-

tions of the Rule, I find it 7 times, which is therefore the other Figure of the Root; which is proved by this, that $2ab + b^2 = 469$, the Resolvend; and because there is no Remainder, the given Number 1369 is a true Square, whose Root is 37.

Example 2. To find the Square Root of 23097636, it is 4806.

Operation.

$$\begin{array}{r}
 23097636 \quad \left(\begin{array}{l} 2bbb \\ 4806 \\ a \end{array} \right. \\
 \underline{a^2 = 16} \\
 2a = 80 \quad \left. \begin{array}{l} 709. \text{ 2d Resolvend.} \\ 704 = 2ab + b^2 \quad (b=8) \end{array} \right) \\
 \underline{(a=40)} \\
 2a = 960 \quad \left. \begin{array}{l} 576. \text{ 3d Resolvend.} \\ 57636. \text{ 4th Resolvend.} \end{array} \right) \\
 \underline{(a=480)} \\
 2a = 9600 \quad \left. \begin{array}{l} 57636 = 2ab + b^2 \quad (b=6) \\ 00000 \end{array} \right) \\
 \underline{(a=4800)}
 \end{array}$$

Explication.

The last Period being 23, the next Square lesser is 16, whose Root is 4, which I place in the Root, and calling it a , I take $a^2 = 16$ out of 23, and to the Remainder 7 I prefix the next Period 90, which makes 709 the 2d Resolvend: Then taking $a = 40$, according to the Rule, I double it, and $2 \times 40 = 80$ is the Divisor, which is contain'd in the Resolvend 8 times; which Number also answers the Limitations of the Rule: For 88×8 is $= 704 = 2ab + b^2$, or $2a + b \times b$, b being 8, wherefore 8 is the next Figure of the Root sought; and sub-

tracting 704 from 709, to the Remainder 5 I prefix the next Period 76, and 576 is the 3d Resolvend; then taking $a = 480$, its Double, 960, is the Divisor; which being greater than the Resolvend, I set 0 in the Root, then prefixing the next Period 36, the 4th Resolvend is 57636, and the Divisor is 9600, (a being here 4800) which is the former Divisor with 0 prefix'd: Then I find 9600 contained in 57636, 6 times, which Number answering the Limitation of the Rule, I set 6 in the Root, and calling it b , I find $2ab + b^2 = 57636$, the Resolvend; so that nothing remains; And so the given Number 23097636 is a perfect Square, whose Root is 4806.

SCHOLIUMS.

1. If you begin your Guess or Trial for the Quote in any of the Steps after the first, at the greatest Number of times, not exceeding 9, that the Divisor is contain'd in the Resolvend: Then the Limitation of the Rule for the Number to be compared with the Resolvend is sufficient to determine when we have the true Figure; because if that Number is greater than the Resolvend the Quote is taken too big; and then we try the next lesser, till it answer the Rule: Yet *observe*, that if you should make trial at random, then tho' $2ab + b^2$ be less than the Resolvend, yet the Quote b may be too little, and you'll prove it by this Mark, *viz.* The Remainder, after taking $2ab + b^2$ out of the Resolvend, may be greater than the Divisor, but it must not exceed the Sum of the Divisor and double the Quote, *i. e.* $2a + 2b$, else the Quote is too little: See below the Explication of Exam. 3. And here I must *observe*, That some Authors think the forming of a Divisor an useless thing, and would have us left altogether to a random Guess for the Figure of the Quote at every Step after the first, tho' they prescribe the same necessary Limitation of the Figure guessed: But they have not considered this Conveniency of the Divisor, that the greatest Number of times it is contained in the Resolvend not exceeding 9, is a Limit to our guessing; for the Figure sought cannot exceed that, and so will in many Cases save the Trouble of guessing at Figures which cannot answer. Besides, the Divisor is of a necessary Consideration in the Demonstration of the Rule; and a further Use of it, see in the next Article.

2. If the Divisor is contained in the Resolvend oftner than 9 times in any Step after the 2d, the Figure sought is certainly 9: And also in the 2d Step it's 9, if the first Figure is at the same time, 5, 6, 7, 8, or 9. But if this is below 5, we must make trial; for sometimes it will be 9, and sometimes not. See Exam. 3, 4.

3. For

3. For forming the Divisor a little more easily, you have no more to do after the 2d Step but add the double of the Figure last found to the last Divisor, and then prefix 0; as you may easily perceive in the Examples.

4. If there is a Remainder after all the Periods are employed, then the given Number is not a perfect Square; and the Root found is the Root of the greatest Integral Square contained in it. How to find a Mixt Root whose Square shall be within any assigned Difference from the given Number, shall be taught in its proper Place.

Example 3. To find the Square Root of 151426. By the Operation we find it's not a Square, but the Root of the greatest Integral Square contained in it is 389.

Operation.

$$\begin{array}{r}
 15\dot{1}4\dot{2}6 \quad (389 \\
 a^2 = 9 \\
 \hline
 2 \times 30 = 60 \quad 614 \\
 \quad 544 = 2ab + b^2 \\
 \hline
 2 \times 380 = 760 \quad 7026 \\
 \quad 6921 = 2ab + b^2 \\
 \hline
 105. \text{ Remainder.}
 \end{array}$$

Explication.

Here in the second Step 60 is contained in 614, 10 times; yet the true Quote, or Figure for the Root, is only 8: For 9 would make the Product 621, which is greater than 614: And had we taken 7, it would have been found too little, from the Mark given in *Schol. 1.*; for then the Product is 499, which taken from 614 leaves 115, which is greater than 60 + 16 (or $2a + 2b = 76$). And because the Remainder of the whole Work is 105, the greatest Square contained in 151426 is 151321, whose Root is 389.

Example 4. To find the Square Root of 15437052: The Root of the greatest Square contained in it is 3929.

Operation.

$$\begin{array}{r}
 15\dot{4}37\dot{0}5\dot{2} \quad (3929 \\
 a^2 = 9 \\
 \hline
 2 \times 30 = 60 \quad 643. \text{ 2d Resolvend.} \\
 \quad 621 = 2ab + b^2. \\
 \hline
 2 \times 390 = 780 \quad 2270. \text{ 3d Resolvend.} \\
 \quad 1564 = 2ab + b^2. \\
 \hline
 2 \times 3920 = 7840 \quad 70652 \text{ 4th Resolvend.} \\
 \quad 70641 = 2ab + b^2. \\
 \hline
 11 \text{ Remainder.}
 \end{array}$$

Explication.

In the second Step, 60 is contained in 643 10 times, and the true Quote is 9. In the third Step, 780 is contained in 2270 only 2 times; and 7840 in 70652 9 times: The Remainder of the Operation being 11. So that the greatest Square contained in the given Number is 15437041, whose Root is 3929.

Demonstration of the preceding Rule.

In order to this Demonstration, the following *Lemma's* must be first demonstrated.

L E M M A I.

The Product of any two Numbers can have at most but as many Places of Figures as are in both the Factors; and at least but one Place fewer. *Exam.* $3 \times 4 = 12$, and $2 \times 16 = 32$. *Demonstr.*

Demonstr. 1. That the Product may have as many Places as both the Factors, one Example is enough to demonstrate. Thus, $46 \times 82 = 3772$; and that in no Case it can have more, I thus prove.

Let any two Numbers be A, B; then take D the least Number possible, which has one Place more than B; it's evident from the Notation of Numbers, that D will consist of 1, with as many o's as the Number of Places in B; and also D will be a greater Number than B; if we then multiply A by D, the Product AD will be equal

$$\begin{aligned} A &= 4678. \quad B = 37549 \\ A \times B &= 175654222 \\ D &= 100000. \quad A \times D = 467800000. \end{aligned}$$

to A, with as many o's before it as are in D, i. e. as the Number of Figures in B: therefore it has as many Places, and can have no more than are in both A and B. But again; since B is a lesser Number than D, therefore AB is a lesser Number than AD, and consequently cannot have more Places, i. e. more than are in A and B both.

2. The Product may have fewer Places than are in A and B both, which one Example will shew. Thus, $23 \times 346 = 7958$: but it can in no Case have above one Place fewer, which is thus proved.

$$\begin{aligned} A &= 23. \quad B = 346 \\ AB &= 7958 \\ D &= 100. \quad AD = 2300. \end{aligned}$$

Take any two Numbers A, B; and take D consisting of 1, with as many o's before it as the Figures less than one in B; i. e. the least Number possible, which has as many Figures as B; then will the Product AD be equal to A, with as many o's before it as are in D, which are one fewer than the Figures in B; consequently AD has as many Places, and can have no fewer than the Sum of the Places in A, and one fewer than are in B; i. e. all the Places in AD can be but one fewer than the Sum of those in A and B. But since B is a greater Number than D, so will AB be greater than AD; and consequently cannot have fewer Places than AD, which can be but one fewer than in A and B both.

COROL. A Number being multiplied into itself, the Product or Square cannot have more Places than double the Places of the Root; and but one fewer at least than that double. Wherefore a Square being distributed into Periods, as the Rule directs, the Root has precisely as many Figures as the Square has Periods.

LEMMA II.

If any Number A is not a Square, yet being distributed into Periods, according to the Rule, the greatest Square contained in it, as N^2 , will have precisely as many Periods as that Number A has.

Exam. 237694 is not a Square, and the greatest Square contained in it is 237169; both which have three Periods.

Demonstr. 1. N^2 cannot have more Periods than A; for then it will have more Figures, and consequently be a greater Number than A; contrary to Supposition.

$$\begin{aligned} A &= 237694 \\ N^2 &= 237169 \\ B &= 10000 \\ B^2 &= 100. \end{aligned}$$

2. Take 1 with as many o's before it as there are Figures standing before the last Period of A (or on the right Hand of it) call the Number arising B; then it is plain that B is a square Number, whose Root is 1, with half as many o's as are in B. For to square any Number expressed by 1 with a Number of o's before it, it's manifest, from the Nature of Multiplication, that the Square is 1, with double as many o's; wherefore B is a Square of as many Periods as A has, and being evidently contained in it, it follows, that the greatest Square contained in it cannot have fewer.

COROL. The Root of the greatest Square, contained in any Number A which is not a Square, hath as many Figures as A has Periods; for it has as many as its own Square has

has Periods (by *Corol. Lemma 1.*) which are as many as A has, by the present *Theorem*.

L E M M A III.

Any Number being distributed into Periods, the greatest Square contained in the last Period on the left, considered as one Number by itself, is the Square of the last Figure of the Root of the given Number, if it is a perfect Square; or of the Root of the greatest Square contained in it, if it's not a Square. Again; the greatest Square contained in the two last Periods, taken as one Number by themselves, is the Square of the last two Figures of the Root of the given Number, or of the greatest Square contained in it; and the same thing is true, comparing the 3 or 4, &c. last Periods, with the Square of the 3 or 4, &c. last Figures of the Root of the given Number, or of the greatest Square contained in it.

Demonstr. Let A be any Number, and B the Square Root thereof, or of the greatest Square contained in it; also let D represent the last, or 2 last, &c. Periods of A , [as in the annex'd Example, take $D = 22$ or 2273 , or 227338] and let r represent the last, or 2 last, &c. Figures of the Root B , [as here 4 or 47, or 476]; so that r^2 is the Square of that last, or 2, &c. last Figures of the Root B .

These things being settled, the Truths to be proved are comprehended in one universal Case, which is this, *viz.* that r^2 in the greatest Square contained in D ; which I shall demonstrate in two Articles: Thus,

$$\begin{array}{r} A = 22733824. \quad B 4768 \\ \quad D \quad \quad \quad r \end{array}$$

1. r^2 is contained in D ; for since (by *Corol. to Lem. 1* and 2.) there are as many Periods in A , as there are Figures in B ; consequently there are, in every Case, as many Periods standing before D in the total A , as there are Figures before r in

the total B ; so that taking D and r in their compleat Values, as they stand in their Totals, there will be as many o's before D , as the Number of Figures in the Periods of A , which stand before, or on the right Hand of D ; and as many o's before r , as the half of those before D . [*Exam.* If $D = 22000000$, then is $r = 4000$; and if $D = 22730000$, then is $r = 4700$.] Then r^2 will have as many o's before it in its compleat Value, as double the Number of o's before the Root r in its compleat Value; and consequently as many as before D . We shall now express these Numbers in their compleat Values; *thus*, 100 , &c. $r^2 00, 00$, &c. $D 0000$, &c. and shew that r^2 is contained in D . For,

If r^2 is greater than D , (both taken without the o's) then is $r^2 0000$, &c. greater than $D 0000$, &c. (r^2 and D being here equally multiplied) by an equal Number of o's prefix'd.) But A is equal to D , with as many Figures before it as there are o's before D or r^2 taken in their compleat Values; (*i. e.* $D 0000$, &c. $r^2 0000$, &c.) Therefore $r^2 0000$, &c. is greater than A ; [for any Figures whatever in the Places of the o's before D , cannot be equal to the Excess of r^2 above D , tho' that Excess were but 1] *i. e.* the Square of 100 , &c. which is but a Part of B , is greater than the Square of B ; because A is at least equal to B^2 : but this is absurd; therefore r^2 cannot be greater than D , and consequently must be contained in it.

2. r^2 is the greatest Square contained in D : For suppose N^2 a greater Number than r^2 ; then take N with as many o's before it, as are before r in its compleat Value, and express it thus, $N 00$, &c. so that its Square is $N 0000$, &c. having double as many o's as $N 00$, &c. the Root has; or as many as $r^2 0000$, &c. or $D 0000$, &c. has. Now because D contains N^2 , by Supposition; therefore $D 0000$, &c. contains $N^2 0000$, &c. Also because N^2 is supposed greater than r^2 ; therefore N is greater than r ; and $N 00$, &c. greater than $r 00$, &c. or than r with as many of any Figures before it; *i. e.* $N 00$, &c. is greater than B , (which is equal to r , with as many certain other Figures before it, as there are o's before

fore

fore r in its compleat Value roo , &c. or in Noo , &c.) so that $Do\dot{o}o\dot{o}$, &c. a Part of A , contains $N^2\dot{o}o\dot{o}$, &c. the Square of Noo , &c. a Number which is greater than B , the Root of the greatest Square contained in A , which is absurd; therefore r^2 is the greatest Square contained in D .

COROL. If we find the Root of the greatest Square contained in the last Period of any Number, we have the last Figure of the Root sought: And if we find the Root of the greatest Square contained in the two last Periods, we have the two last Figures of the Root sought, and so on; which so far explains the Investigation of the Rule; what remains to compleat it, you have in the following

L E M M A IV.

Part 1. If the Root of any known Square is supposed to consist of two Parts, or Members; then if one of these Members is known, we have a Rule for finding the other from the Consideration of the Square of a Binomial Root. Thus: If the Root is $A + B$, the Square is $A^2 + 2AB + B^2$, viz. the Sum of the Squares of the two Parts, and twice the Product of these Parts; wherein it is evident, that if the Square of either Part, as A^2 , is subtracted from the total Square $A^2 + 2AB + B^2$, the Remainder is the Sum of the Square of the other Member, and the double Product of the two Members, viz. $2AB + B^2$. Now suppose A to be known; if we take $2A$ for a Divisor, and find how oft it is contained in that Remainder; but under this Limitation, viz. that the Quote being added to the Divisor, and the Sum multiplied by the Quote, the Product shall be equal to the Dividend $2AB + B^2$; then it is manifest, that the Quote can be no other Number than B , the other Member of the Root sought. For since $2A + B \times B = 2AB + B^2$ the Dividend, therefore it's plain that no other Number but B added to $2A$, and the Sum multiplied by the same B , will produce $2AB + B^2$; since either a greater or lesser Number added to $2A$, makes a greater or lesser Sum; which being multiplied by the same Number, produces still a greater or lesser Number.

Part 2. Tho' a Number is not a Square, yet having one Member of the Root of the greatest Square contained in it, we can find the other Member by the same Method, as if it were a Square. Thus:

$$\begin{aligned} M &= A^2 + 2AB + B^2 + R \\ M - A^2 &= 2AB + B^2 + R = D \end{aligned}$$

Let M be any Number not a Square, and $A + B$ the Root of the greatest Square contained in it; the Square is therefore $A^2 + 2AB + B^2$. Also let

R be the Number that's more than $A + B$ in M , so that $M = A^2 + 2AB + B^2 + R$. Now A being known, if we take A^2 from M , the Remainder is plainly $2AB + B^2 + R$, which we may call D . And if we find how oft $2A$ is contained in D under this Limitation, viz. that the Quote being added to the Divisor, and the Sum multiplied by the same Quote, the Product shall still be less than D : [For this is to be observed, that there is no Number which will make a Product equal to D ; because then M would be a Square; therefore any Number you can take, will make the Product either greater or lesser than D .] Then, I say, the Quote is the other Member of the Root sought, viz. B : For let us suppose the Quote is another Number N , then if N is less than B , it follows, contrary to Supposition, that N is not the greatest Number qualified according to the Rule, viz. which added to the Divisor, and the Sum multiplied by the same Number, makes a Product less than D ; for B is greater than N , and yet is a Number so qualified, because $D = 2AB + B^2 + R = 2A + B \times B + R$: Therefore N is not less than B . Nor, again, can it be greater; for by Supposition $2A + N \times N (= 2AN + N^2)$ is less than $D (= M - A^2)$ and adding A^2 to both, then $A^2 + 2AN + N^2 (= A + N)^2$ is less than M , and is therefore contained in it. But again; since N is greater than

than B , $A + N$ is greater than $A + B$, and $\overline{A + N}^2 (= A^2 + 2AN + N^2)$ greater than $\overline{A + B}^2 (= A^2 + 2AB + B^2)$ and consequently this is not the greatest Square contained in M , as was supposed: Wherefore N is not greater than B ; and if it's neither greater nor lesser, it must be equal.

COROLL. If a given Number M is not a Square, the Number R which is over the greatest Square contained in it, (and is necessarily the Remainder, which happens in the Operation after B the second Member of the Root is found) may be greater than $2A$ the Divisor; because we have not taken $2A$ out of the Dividend D as oft as possible; but it can never exceed double of the Root found, if that is the true Root of the greatest Square contained in M : For let the Root found be called N , if the Remainder exceeds $2N$, it must be at least $2N + 1$, and if this is added to N^2 , the Sum $N^2 + 2N + 1$, $(= \overline{N + 1}^2)$ being evidently contained in M , it follows that N is not the Root of the greatest Square contained in it, as was supposed.

APPLICATION of the preceding Lemma's for demonstrating the Extraction of the Square Root.

1. The first and second *Lemma's* are already applied; from whence are deduced, as *Corollaries*, the first Thing asserted in the *Rule*, viz. That the Root must have as many Figures as the given Number has Periods.

2. From *Lemma III.* we have the Reason why the given Number is pointed from the Right Hand to the Left; because, being done so, it is demonstrated that the last Figure of the Root sought, the two last, and so on, make the Root of the greatest Square contained in the last, the two last, &c. Periods of the given Number.

3. The remaining Part of the Rule is to find the Figures of the Root, one after another, out of these Periods; the Reason of which is contained in *Lemma III.* and *IV.* and its *Coroll.* and is deduced thus:

We first take the last Period, and the greatest Square contained in it we seek in the Table of simple Squares, [which must be found there; for since a Period has but two Figures at most, the Root of the greatest Square contained in it can be but one Figure; because the Square of 10, the least Number of two Figures, is 100, which has three Figures]. The Root of this Square is, by *Lem. 3.* the last Figure of the Root sought. So in the preceding *Example 3.* the given Number is 151426; the last Period is 15, and the greatest Square contained in it is 9, whose Root is 3, the last Figure of the Root sought.

Now if we suppose the two last Periods 1514 to be the given Number, then the Root of the greatest Square contained in it has but two Figures, whereof we have found the last, viz. 3, whose Real Value is 30; and to find the other, it's plain, from *Lemma IV.* that calling the first Member of the Root now found, viz. $30 = a$, and calling the Member sought b , then the greatest Square contained in 1514 is $a^2 + 2ab + b^2$; but $a^2 = 900$, or rather 900 taken in its true Value; so that 9 from 15, and 14 prefix'd to the Remainder, (which is the Method of the Rule) is the same thing as 900 from 1514: The Remainder is 614, the second Resolvend, which is equal to $2ab + b^2$ at least, with some Remainder over perhaps; we shall therefore call the Remainder $2ab + b^2 + r$: What remains then, is to find this second Member of the Root b ; and according to *Lem. IV.* if we make 22 the Divisor, and find how oft it is contained in the Resolvend $2ab + b^2 + r$, so that calling the Quote b , this Quote added to the Divisor, and the Sum $(2a + b)$ multiplied by b , the Product $(2ab + b^2)$ shall not exceed the Resolvend $(2ab + b^2 + r)$; then it's shewn that the Quote is truly the second Member of the Root: But it's manifest that this is the very Method of the Rule; wherefore it's just and good

good when the Root sought has but two Figures. Again; The Number given having three Periods, as if it were 151426, then having found 38 the Root of the greatest Square contained in the two first Periods 1514, (as already shewn); these are the two last Figures of the Root of 151426, (by *Lem. 3.*) And if we take 38 in its true Value it is 380, because there is another Figure on its Right in the Root sought; then 380 being considered as one Member of the Root sought, we call it also a ; and by *Lemma IV.* we are to subtract its Square, viz. $\overline{300} + \overline{80}^2 = \overline{300}^2 + 2 \times 300 \times 80 + \overline{80}^2$ out of the given Number 151426: But this is already done, because we have taken first the Square of 3, (which was in the former Step called a) viz. 9, out of 15, which is equivalent to taking the Square of 300, (which is now a) viz. 90000, out of 151426, which leaves 61426; then b being 8, and $a = 30$, we have taken $2ab + b^2 = 544$ out of 614, the former Remainder; to the Remainder 70 we have prefix'd 26, the first Period, which makes the whole 7026; and this is the same Number which remains, if taking $b = 80$, and $a = 300$, we take $2ab + b^2 = 48000 + 6400 = 54400$ out of the former Remainder 61426. Now the Square of 380 being taken out of the given Number 151426, the Remainder 7026 is the next Refolvend; and for a Divisor we have made $2 \times a = 2 \times 380 = 760$, and the Member sought we have found the same way as before, which is both according to *Lemma IV.* and the Rule for Extraction; which is therefore good for any Root of three Figures.

If there are more than three Figures in any Root, the Reasons of the Rule from one Step to another for ever are manifestly the same, and need not be further insisted on. I shall only illustrate this Application by one Example of a perfect Square, whose Involution by the Method shewn in the preceding Section, and its Evolution by the present Rule, will illustrate one another; and you'll evidently perceive, that as by knowing the true Place of every Figure found in the Root, we may take it in its compleat Value, and perform the Work that way, as in the following Operation; yet we save the trouble of many superfluous Figures by the Method of the Rule, which produces the same Effect.

Involution of 389 to its Square makes
151321.

Thus:

$$\begin{array}{l}
 \text{Root.} \\
 389 = 300 + 80 + 9 \\
 \hline
 90000 = a^2. (a = 300) \\
 48000 = 2ab (b = 80) \\
 6400 = b^2 \\
 \hline
 144400 = a + b^2 = 380^2 = A^2 \\
 6840 = 2AB, (B = 9) \\
 81 = B^2 \\
 \hline
 151321 = A + B^2 = 380 + 9^2 = 389^2
 \end{array}$$

Evolution of 151321 to its Square Root
makes 389.

Thus:

$$\begin{array}{rcl}
 & \text{A} & \text{B} \\
 & \text{---} & \text{---} \\
 \text{Square.} & a & b \\
 151321 & (300 + 80 + 9) & \\
 a^2 = 90000 & & \\
 \text{Divisor. } 2 \times 300 & \text{61321 Refolvend.} & \\
 \hline
 48000 = 2ab & & \\
 6400 = b^2 & & \\
 \hline
 54400 = 2ab + b^2 & & \\
 \hline
 \text{Divisor. } 2 \times 380 & \text{6921 Refolvend.} & \\
 \hline
 6840 = AB & & \\
 81 = B^2 & & \\
 \hline
 6921 = 2AB + B^2 & & \\
 \hline
 0000 & & \\
 \hline
 \end{array}$$

Thele

These Operations are reverse of one another; and as to the Evolution, it differs from the Method of the Rule in this only, that the several Members of the Root are here written in their compleat Value, which occasions the writing down many Figures unnecessarily, which we avoid the other Way.

There remain yet two things to be demonstrated, which are delivered in *Schol.* 1. and 2.

1st. The Remainder, after every Figure of the Root is found, cannot exceed the Sum of the Divisor and double the Quote: The Reason of this is contained in *Corol. Lemma IV.* where it's shewn, that what's over the greatest Square contained in any Number cannot exceed double the Root of the greatest Square; which is plainly the Sum of the Divisor at every Step, and double the Quote; for the Divisor is double of all the preceding Figures taken in their compleat Value, which therefore added to double the Quote, makes double all the Root. Thus, if any Root is $a + b$, the Divisor for finding b is $2a$; and when $2ab + b^2$ is taken out of the Resolvend, call the Remainder r ; and in the *Coroll.* to *Lemma IV.* it's shewn that r cannot exceed $2a + 2b$.

2. If the Divisor is contained oftner than 9 times in the Resolvend, after the second Step, or after the second Figure of the Root is found, the Figure sought is 9. For since the Resolvend contains the Divisor ($2a$) at least 10 times, it may be represented thus; $2a \times 9 + 2a + R$. Now taking 9 for the Quote, the Product according to the Rule is $2a \times 9 + 9 \times 9$, which cannot exceed the Resolvend, because 9×9 cannot exceed $2a$, which in this Case exceeds 100, since there being two Figures found in the Root, and 0 prefix'd to them in order to form the Divisor $2a$, then is $2a$ a Number of at least three Places, which is therefore greater than $9 \times 9 = 81$. Lastly, since it is demonstrated that in every Step, the Quote, under the Limitation of the Rule, is but one Figure; and 9, which is the greatest Number of one Figure, makes a Product not exceeding the Resolvend; therefore 9 is the Number sought.

In the second Step, if the Divisor is oftner than 9 times contained in the Resolvend, then it's plain, that if 81 is less than $2a$, [as it will certainly be when a is 5, 6, 7, 8, or 9, i. e. 50, 60, 70, 80, or 90, as they are taken in forming the Divisor; for then the Doubles, or $2a$, are 100, 120, 140, 160, 180.] then 9 is the Figure sought; because $2a \times 9 + 81$ must be less than $2a \times 9 + 2a + R$, the Resolvend, since 81 is less than $2a$. But if the first Figure is 1, 2, 3, or 4, that is, if $2a$ is 2×10 , 2×20 , 2×30 , or 2×40 , i. e. 20, 40, 60, or 80, which are less than 81, then the Figure sought will be less than 9, if $2a + R$ is less than 81; and it will be 9, if $2a + R$ is equal to or greater than 81: Because the Resolvend being $2a \times 9 + 2a + R$, if the Figure sought is made 9, then the thing to be subtracted from the Resolvend is $2a \times 9 + 81$, so that $2a + R$ must be at least equal to 81; and if it is not, we must therefore take a less Figure for the Quote, so that the Resolvend be at least equal to the Number to be subtracted.

Of the Proof of the Square Root.

As Extraction is opposite to the Raising of Powers, so the one is the Proof of the other: Thus; To prove the Square Root, multiply it by itself, and if the Product is equal to the given Square, or to the given Number after the Remainder of the Extraction is taken out of it, then the Extraction is right done.

But this may be also proved by casting out of 9's: Thus; Cast the 9's out of the given Number, if there is no Remainder in the Extraction; or out of the Difference of that Number and Remainder; then cast the 9's out of the Root found, and multiply the Excess (or what it wants of 9) by itself, and cast the 9's out of the Product; if the Excess, or what it wants of 9, is equal to the preceding Excess of 9's, the Extraction is right.

Example 1. The Square Root 256 is 16; proved thus; the Excess of 9's in 256 is 4. In 16 it is 7, and this multiplied by itself is 49, in which the Excess of 9's is also 4.

Example 2. The greatest Square Root contained in 230 is 15, and 6 remains: For $230 - 6 = 225$, in which there is 0 over 9's; then in 15 there is 6 over 9; and $6 \times 6 = 36$, in which there is also 0 over 9's.

The Reason of this Practice is evident from what is demonstrated of it in Multiplication.

Problem II. To Extract the Cube Root of a Whole Number.

Roots.	Cubes.
1 :	1
2 :	8
3 :	27
4 :	64
5 :	125
6 :	216
7 :	343
8 :	512
9 :	729

RULE I. **M**AKE a Table of simple Cubes, as in the Margin; then

II. Distribute the given Number into Periods of three Figures, beginning at the Right Hand: The Number of Periods shews the Number of Figures in the Root.

III. Begin at the last Period, which is the first Resolvend, and seek it or the next Cube Number less than it in the Table of simple Cubes, the Root of that is the last Figure (or that in the highest Place) of the Root sought; which being set down, subtract its Cube out of the last Period; to the Remainder prefix the next Period, and you have the next Resolvend.

IV. Consider the Figure found as in the Place of 10's, or with 0 prefix'd, and under that Value take the Triple of it, and also the Triple of its Square, making the Sum of these the Divisor; [which being composed of two Parts, it will be convenient to distinguish them by calling the Triple Square the first Part, and the other the second Part.] Then find how oft this Divisor is contained in the Resolvend last formed; which must never be taken above 9, (tho' it may be oftner contained) and then also it must be under this Limitation, *viz.* That having multiplied the first Part of the Divisor by the Quote, (now found) and the second Part by the Square of that Quote; and, lastly, to the Sum of these two Products adding the Cube of the Figure found; this Sum shall not exceed the Resolvend: which Sum being therefore subtracted out of the Resolvend, and the next Period prefix'd to the Remainder, you have the next Resolvend.

V. Take both the Figures of the Root already found, and considering them as so many 10's, *i. e.* place 0 before them, and under that Value take the Triple of that Number, and also the Triple of its Square; whose Sum is your next Divisor, distinguished into first and second Part, as before: Then find how oft this Divisor is contained in the Resolvend last formed, under the same Limitations as before; place the Figure found on the Right of these already found in the Root; and subtracting from the Resolvend, as formerly directed, to the Remainder prefix the next Period for a new Resolvend.

VI. Make a new Divisor from the Figures of the Root found in the same manner as in the preceding Steps, and divide, under the same Limitation; and thus proceed till all the Periods are taken in; finding at every Step a new Figure of the Root: which will in some Cases be 0, as when the Divisor is greater than the Resolvend, or when 1 added to the Divisor makes the Sum greater than the Resolvend; in which Case, after the 0 is set in the Root, prefix the next Period to the same Resolvend; and go on, forming a new Divisor by prefixing another 0 to the last Divisor.

SCHOLIUMS.

These *particular* OBSERVATIONS may be usefully added to this Rule (tho' it's a compleat *general* Rule by itself):

1. If you begin your Trials for the Quote at the greatest Number of Times (not exceeding 9) that the Divisor is contained in the Resolvend; then the Limitation of the Rule, for the Number to be compared with the Resolvend, is sufficient to determine when you have the true Figure. Yet it will be useful to *observe*, That if you should begin at a lesser Figure than the Remainder, tho' it may be greater than the Divisor, yet it must never exceed the Sum of these two Numbers, *viz.* Triple all the Figures of the Root already found, (taken as one Number) and triple its Square.

2. If the Divisor is contained in the Resolvend oftner than 9 times, at the second Step, or when you seek the second Figure of the Root; and if, at the same time, the first Figure is 8 or 9, then the Figure sought is certainly 9: But if the first Figure is less than 8, you must make Trials. Again; If in any Step after the second, the Divisor is oftner than 9 times contained in the Resolvend, the Figure sought is certainly 9.

3. If there is a Remainder after all the Periods are employ'd, the given Number is not a Cube; and the Root found is that of the greatest integral Cube contained in it. How to find a Mixt Root whose Cube shall be within any assigned Difference from the given Number, shall be taught in its proper Place.

EXAMPLES.

Example 1. The Cube Root of 614125 is 85.

Operation.

Explication for Exam. 1.

$$\begin{array}{r}
 614125 \quad (a.b) \\
 \underline{a^3 = 512} \\
 102125 \quad \text{2d Resolvend.} \\
 \underline{96000 = 3a^2 \times b} \\
 \quad 6000 = 3a \times b^2 \\
 \quad \quad 125 = b^3 \\
 \hline
 102125 \quad \text{Sum.} \\
 \hline
 000000 \quad \text{Remainder.}
 \end{array}$$

The given Number being pointed, the last Period is 614, and the next Cube to that is 512, whose Root is 8, which I call a , and subtracting $a^3 = 512$ from 614, the Remainder is 102, to which the next Period prefix'd makes 102125, the 2d Resolv. Then for a Divisor I take $a = 80$, and so $a^2 = 6400$, and $3a^2 = 19200$; then $3a = 240$, and $3a^2 + 3a = 19200 + 240 = 19440$, the Divisor; which is contained in the Resolvend, under the Limitation of the Rule, 5 times, the 2d Figure of the Root, which calling b , then $3a^2 \times b = 96000$, $3a \times b^2 = 6000$, $b^3 = 125$, and the Sum of these is 102125, equal to the Resolvend; so that the given Number is a true Cube, whose Root is 85.

Example

Example 2. The Cube Root of 41421736 is 346.

Operation.

$$\begin{array}{r}
 41421736 \quad \begin{array}{l} (ab \\ 346 \\ AB \end{array} \\
 a^3 = 27 \\
 \hline
 3a^2 + 3a = 2790 \quad \begin{array}{l} (a = 30) \\ (a^2 = 900) \end{array} \quad \begin{array}{l} 14421. \text{ 2d Resolvend.} \\ 10800 = 3a^2 \times b \\ 1440 = 3a \times b^2 \\ 64 = b^3 \\ \hline 12304 \text{ Sum.} \end{array} \\
 \hline
 3A^2 + 3A = 347820 \quad \begin{array}{l} (A = 340) \\ (A^2 = 115600) \end{array} \quad \begin{array}{l} 2117736. \text{ 3d Resolvend.} \\ 2080800 = 3A^2 \times B \\ 36720 = 3A \times B^2 \\ 216 = B^3 \\ \hline 2117736 \text{ Sum.} \\ \hline 0000000 \text{ Remainder.} \end{array}
 \end{array}$$

form the Divisor, I take $A = 340$, whence $A^2 = 115600$; and the Divisor is 347820, whence the last Figure of the Root is 6, as the Work shews.

Example 3. If we seek the Cube Root of 705919947284, we find it's not a perfect Cube, but the Root of the greatest Cube contained in it is 8904; whose Cube is 705919947264, and the Remainder is 20, as you see in the Operation.

Operation.

$$\begin{array}{r}
 705919947284 \quad \begin{array}{l} (8904 \\ 8: = 512 \end{array} \\
 \hline
 \text{Divisor} = 3a^2 + 3a = 19440 \quad \begin{array}{l} (a = 80, a^2 = 6400) \end{array} \quad \begin{array}{l} 193919. \text{ 2d Resolvend.} \\ 172800 = 3a^2 \times b \\ 19440 = 3a \times b^2 \\ 729 = b^3 \\ \hline 192969 \text{ Sum.} \end{array} \\
 \hline
 \begin{array}{l} (a = 890) 3a^2 + 3a = 2378970 \\ a = 8900 3a^2 + 3a = 237656700 \end{array} \quad \begin{array}{l} 950947. \text{ 3d Resolvend.} \\ 950147284. \text{ 4th Resolvend.} \\ 950520000 = 3a^2 \times b \\ 387200 = 3a \times b^2 \\ 64 = b^3 \\ \hline 950907264. \text{ Sum.} \\ \hline 20. \text{ Remainder.} \end{array}
 \end{array}$$

Here the 3d Resolvend being less than the Divisor, I put 0 in the Root, and form a new Resolvend by the next Period. The rest of the Work is obvious. DE

DEMONSTRATION of the preceding Rule.

LEMMA I.

If any three Numbers are multiplied into one another, the Product can have at most but as many Figures as are in all the three Factors, and at least but two fewer. Example: $3 \times 4 \times 9 = 108$.

Demonstr. This is a plain Consequence of *Lem. I.* for the Square Root: Because the Product of two Factors cannot have more Places than are in both Factors, or but one fewer at least; which Product being considered as one Factor, and multiplied by a third Factor, the same is true of this new Product; which makes the Truth proposed manifest.

COROLL. The Cube of any Number can have at most but as many Figures as triple the number of Figures in the Root, and but two fewer at least. Wherefore, again; Any Cube being distributed into Periods of three Places, the Number of Periods, and the Number of Figures in the Root must necessarily be equal; and the last Period may, in some Cases, consist only of one or two Figures.

LEMMA II.

If any Number A is not a Cube, yet being distributed into Periods, according to the preceding Rule, the greatest Cube contained in it, as N^3 , will have precisely as many Periods as that Number A has.

Example: 35987 is not a Cube; but being pointed has two Periods, viz. as many as the greatest Cube contained in it, 35937.

$A = 35987$ $N^3 = 35937$ $B = 1000$ $B^{\frac{1}{3}} = 10$	<p><i>Demonstration I.</i> N^3 cannot have more Periods than A, for then it will have more Figures, and consequently be a greater Number than A, contrary to Supposition.</p> <p>2. Take 1 with as many o's before it as there are Figures standing before the last Period of A; (i.e. on the right Hand of it) call the Number arising B: Then it's plain that is a Cube Number, whose Root is 1, with as many o's before it as $\frac{1}{3}$ Part of the Number of o's before the 1 in the Cube B; for to cube any Number expressed by 1 with o's, it's manifest from the Nature of Multiplication, that the Cube is 1 with three times as many o's; wherefore B is a Cube of as many Periods as A has; and being evidently contained in it, it follows, that the greatest Cube contained in it cannot have fewer.</p>
--	--

COROLL. The Root of the greatest Cube contained in any Number A, which is not a perfect Cube, hath as many Figures as A hath Periods: For it hath as many as its own Cube hath Periods, (by *Coroll. Lemma I.*) which are as many as A has, by the present *Theorem*.

LEMMA III.

Any Number being distributed into Periods according to the Rule, the greatest Cube contained in the last Period on the left, consider'd as one Number by itself, is the Cube of the last Figure of the Root of the given Number, if it's a perfect Cube; or of the Root of the greatest Cube contained in it, if it's not a Cube. Again; the greatest Cube contained in the two last Periods, taken as one Number by themselves, is the Cube of the last two Figures of the Root of the given Number, or of the greatest Cube contained in it. And the same thing is true, comparing the 3 last or 4 last Periods, and so on, with the

Cube

Cube of the 3 or 4, &c. last Figures of the Root of the given Number, or of the greatest Cube contained in it.

Demonstr. Let A be any Number, and B the Cube Root thereof, or of the greatest Cube contained in it. Also let D be the last, or 2 last, or 3 last, &c. Periods of A; [as in the annex'd Example, take $D=41$, or 41421] and let r represent the last, or two last, or three last, &c. Figures of the Root B, (as here 3 or 34) so that r^3 is the Cube of that last or two last, &c. Figures of the Root. These things being settled, the Truths to be proved are comprehended in one universal Case, thus; *viz.* r^3 is the greatest Cube contained in D; which I shall demonstrate in two Articles, thus:

$A = \begin{matrix} 41421736 \\ D \end{matrix} \} B = 346.$

Figures before r in the total B ; so that taking D and r in their compleat Value, as they stand in their Totals, there will be as many o's before D , as the Number of Figures of the Periods in A , which stand before, or on the right Hand of D ; and as many o's before r , as the third Part of these before D . [For Example: If $D = 41,000,000$, then is $r = 300$. If $D = 41421,000$, then is $r = 340$.] Then r^3 will have as many o's before it, in its compleat Value, as triple the Number of o's before the Root r in its compleat Value, and consequently as many as are before D . We shall now express these Numbers in their compleat Values, thus, 100 , &c. $r^3 = 000000$, &c. $D = 0000,000$, &c.

Again; If r^3 is greater than D, then is $r^3000,000, \&c.$ greater than $D000,000, \&c.$ (being equally multiplied by the equal Number of o's prefix'd.) But A is equal to D, with as many certain Figures before it as there are o's before D, or r^3 in their compleat Values, *i. e.* $D000,000, \&c.$ or $r^3000,000, \&c.$ therefore $r^3000,000, \&c.$ is greater than A; [for any Figures whatever in the Places of the o's before D, cannot be equal to the Excess of r^3 above D, tho' that Excess were but 1]; *i. e.* the Cube of 100, *&c.* which is but a Part of B, is greater than the Cube of B, because A is at least equal to B^3 . But this is absurd; therefore r^3 cannot be greater than D, and consequently must be contained in it.

2. r^3 is the greatest Cube contained in D. For suppose N^3 a greater Number than r^3 ; then take N with as many o's before it, as are before r in its compleat Value, and express it thus, $N00, \&c.$ so that its Cube is $N000,000, \&c.$ having triple as many o's as $N00, \&c.$ the Root has; or as many as $r^3000,000, \&c.$ or $D000,000, \&c.$ has. Now because D contains N^3 (by Supposition) therefore $D000,000, \&c.$ contains $N^3000,000, \&c.$ Also, because N^3 is supposed greater than r^3 , therefore N is greater than r , and $N00, \&c.$ greater than 100, *&c.* or than r with as many of any Figures before it; *i. e.* $N00, \&c.$ is greater than B, (which is equal to r with as many certain other Figures before it, as there are o's before r in its compleat Value 100, *&c.* or in $N00, \&c.$) so that $D000,000, \&c.$ a Part of A contains $N^3000,000, \&c.$ the Cube of $N00, \&c.$ a Number which is greater than B, the Root of the greatest Cube contained in A, which is absurd; therefore r^3 is the greatest Cube contained in D.

COROL. If we find the Root of the greatest Cube contained in the last Period of any Number, we have the last Figure of the Root sought; and if we find the Root of the greatest Cube contained in the two last Periods, we have the two last Figures of the Root sought, and so on. Which so far explains the Investigation of the Rule; what remains to compleat it, you have in the following

LEMMA IV.

Part I. If the Root of any known Cube is supposed to consist of two Parts; then if one of these Parts is known, we can find the other by means of the Cube of a Binomial Root. Thus:

$\overline{A+B^3} = A^3 + 3A^2B + 3AB^2 + B^3$; wherein it's evident, that if the Cube of the known Part A , *viz.* A^3 , is subtracted from the Cube of the whole, the Remainder is $3A^2B + 3AB^2 + B^3$. Now since A is known, so also is $3A^2 + 3A$; and if we seek how oft this is contained in the preceding Remainder, under this Limitation, that the first Member of the Divisor, $3A^2$, being multiplied by the Quote, and the second Member $3A$ being multiplied by the Square of the Quote, and to these two Products the Cube of the Quote be added, the Sum shall be equal to the Dividend; then the Quote shall be equal to B the Number sought; because no other Number but B can answer to this Condition: For if you call the Quote D , then must $3A^2D + 3AD^2 + D^3$ be equal to $3A^2B + 3AB^2 + B^3$; which is manifestly impossible, unless $D = B$; since otherwise the respective Members of the one will be lesser or greater than those of the other, and consequently the Wholes will not be equal.

Part 2. Tho' a Number is not a Cube, yet having one Member of the Root of the greatest Cube contained in it, we can find the other by the same Method, as if it were a Cube. Which will easily appear. *Thus*:

$M = A^3 + 3A^2B + 3AB^2 + B^3 + R$ | Let M be a Number, not a Cube;
 $M - A^3 = 3A^2B + 3AB^2 + B^3 + R = D$ | and $A + B$ the Root of the greatest
 Cube contained in it; which Cube is therefore $A^3 + 3A^2B + 3AB^2 + B^3$. Again; let
 R be the Number that's more than that Cube in M ; so that $M = A^3 + 3A^2B + 3AB^2 + B^3 + R$. Now A being known, take A^3 from M , the Remainder is $3A^2B + 3AB^2 + B^3 + R$; which we may call D : And then if we find how oft $3A^2 + 3A$ is contained in D , under these Limitations, *viz.* that the Quote being multiplied into $3A^2$, and the Square of the Quote multiplied into $3A$, and to these Products the Cube of the Quote be added, the Sum shall still be less than D . [For *observe*, that whatever Number you chuse for the Quote, it will make this Sum either greater or lesser than D , and never equal; because were it equal, then M would be a Cube, contrary to Supposition.] Then, I say, the Quote is equal to B , the other Member of the Root sought. Because, if it can be different, suppose it to be N ; which is either lesser or greater than D : But it cannot be lesser; for then it would follow, that, contrary to Supposition, N is not the greatest Number qualified according to the Rule, *viz.* so that $3A^2N + 3AN^2 + N^3$ is less than D ; for B is greater than N , and yet is so qualified, since $D = 3A^2B + 3AB^2 + B^3 + R$: Wherefore N cannot be less than B ; nor can it be greater, because, by Supposition, $3A^2N + 3AN^2 + N^3$ is less than $D (= M - A^3)$; and adding A^3 to both, then $A^3 + 3A^2N + 3AN^2 + N^3 (= \overline{A+N^3})$ is less than M , and therefore is contained in it. But again; $A + N$ is greater than $A + B$, and $\overline{A+N^3}$ greater than $\overline{A+B^3}$, consequently $\overline{A+B^3}$ is not the greatest Cube contained in M , contrary to Supposition; so that N cannot be greater than B : Wherefore, lastly, since N cannot be either lesser or greater than B , it must be equal to it.

COROL. If a Number M is not a Cube, the Number R , which is over the greatest Cube contained in it, (which is necessarily the Remainder after the second Member B is found) can never exceed the Sum of triple the Root found, and triple its Square: For if the Root found is N , then if the Remainder exceed $3N^2 + 3N$, it must be at least $3N^2 + 3N + 1$; which added to N^3 makes $N^3 + 3N^2 + 3N + 1 = \overline{N+1^3}$. And since this Cube is manifestly contained in M , (for it's the Sum of the greatest Cube N^3 contained in M , and the Remainder $3N^2 + 3N + 1$ added); it follows, contrary to Supposition, that N is not the Root of the greatest Cube contained in M , because $N + 1$ is greater than N ; and $\overline{N+1^3}$ is contained in M , if R is greater than $3N^2 + 3N$; therefore this cannot be.

APPLICATION of the preceding Lemma's for demonstrating the Extracting of the Cube Root.

1. The first and second *Lemma's* are already applied; from whence are deduced as *Corollaries*, the first thing asserted in the Rule, *viz.* That the Root must have as many Figures as the given Number has Periods.

2. From *Lem. 3.* we have the Reason why the given Number is pointed from the right Hand, *viz.* because, being done so, it is demonstrated, that the last Figure of the Root sought, (*i. e.* the Figure in the highest Place) the two last, &c. make the Root of the greatest Cube contained in the last, or two last, &c. Periods of the given Number.

3. The remaining Part of the Rule is to find the Figures of the Root, one after another, out of these Periods; the Reason of which is contained in *Lem. 3* and *4*, and its *Corol.* and is deduced thus:

We first take the last Period; and in the Table of simple Cubes, we seek that Number, or the next Letter, whose Root is, by *Lem. 3.* the highest Figure of the Root sought. So in the preceding *Example*, the given Number is 41421736, which we shall here call N. The last Period is 41, and the next Cube to this is 27, whose Root is 3, the last Figure of the Root sought. Now if we suppose the two last Periods 41421 to be the whole of the given Number, then the Root of the greatest Cube contained in it has but two Figures, whereof we have now found the last; and to find the other (which is the next Figure of the Root sought, by *Lem. 3.*) we proceed thus: Calling the Figure found *a*, we subtract its Cube $a^3 = 27$, from 41 the last Period; and to the Remainder 14 prefixing the next Period 421, the whole 14421 is the 2d Resolvend. And observe, that as the 41 is really 41000, in respect of the total 41421; so the Figure found is really 30, in respect of the next to be found; and in that Value we do actually take it by subtracting it from 41, considering where this stands, and which the prefixing the next Period to the Remainder does farther clear: For this is the same thing as if we had written 27000 the Cube of 30, and taken that from 41421; wherefore this is the same Operation as that explained in *Lem. 4.* *i. e.* having found 30 the first Member of the Root of 41421, we take its Cube out of the whole; and out of the Remainder 14421, we seek the next Member of the Root, which we know cannot exceed 9, because it's the second Member of a Root consisting of two Figures; whereof we have found that belonging to the highest Place, which consider'd in its compleat Value is the first Member. Now to find the Figure sought, we form a Divisor according to the Rule (demonstrated in *Lem. 4.*) thus: Taking $30 = a$, the Divisor is $3a^2 + 3a = 3 \times 900 + 3 \times 30 = 2700 + 90 = 2790$. And this we find contained in the Resolvend 14421, 5 times; but under the Limitation of the Rule we can take it at most 4 times; and 4 is the Figure sought; which calling *b*, the Proof of its being the true Figure is this: We take $3a^2b + 3ab^2 + b^3 = 12304$, which is less than the Resolvend 14421; and 4 is therefore the right Figure, because 5 would have made $3a^2b + 3ab^2 + b^3$ greater than 14421. Or had we at a guess taken $3 = b$, then would it be $3a^2b + 3ab^2 + b^3 = 8937$; which taken from 14421 leaves 5484; which is greater than $3 \times 33^2 + 3 \times 33 = 3 \times 1089 + 3 \times 33 = 3267 + 99 = 3366$; and therefore 3 is too little, as is shewn in *Schol. 1.* added to the Rule. Thus we have found 34, the Root of the greatest Cube contained in 41421, (the Remainder, or what is over, being 2117) and have shewn that the Rule is just and good for a Root of two Figures. Again; For a Number of three Periods, as 41421736, whose Root has three Figures; having found the two Figures in the highest Places; and taking these with 0 prefix'd, which makes the true Value; and calling this again *a*, or *A*, the first Member of the Root, the second Member, which is a single Figure, is found the same way as before explained; which is according to the Rule. But now in this there is so much of the Work already done; for the Cube of

this first Member, or A^3 , is already subtracted from the Total 41421736, because A is now equal to the former $a + b$; and it's evident from the Work that we have subtracted a^3 , and then $3a^2b + 3ab^2 + b^3$ to make the Cube of $a + b$. It's true, we have taken $a = 30$ and $b = 4$; whereas $A = 380$, so that a should be 300 and $b = 80$: But by the Places in which we have set a^3 , and $3a^2b + 3ab^2 + b^3$, we have in effect taken them, as if it had been $a = 300$ and $b = 80$; and so we have duly subtracted A^3 , or the Cube of 380, from the Total 41421736; the Remainder whereof is 2117, to which the next Period 736 is prefix'd, making 2117736 the Resolvend for finding the next Figure; which we find to be 6, by the same Rule and Reason as we found the last Figure.

If there are more than three Figures in any Root, the Reasons of the Rule are manifestly the same from one Step to another *in infinitum*. I shall add for an Illustration one Example, wherein each Figure of the Root is taken in its compleat Value.

Involution of 346 to its Cube, makes 41421736.

Thus:

$$\begin{array}{l} \text{Root} \\ 346 = 300 + 40 + 6 \\ \hline 27000000 = a^3. (a = 300.) \\ 10800000 = 3a^2b. (b = 40.) \\ 1440000 = 3b^2a. \\ 64000 = b^3. \\ \hline 39304000 = a + b^3 = A^3. \\ 2080800 = 3A^2B. (A = 340. B = 6.) \\ 36720 = 3AB^2. \\ 216 = B^3. \\ \hline 41421736 = A + B^3. \end{array}$$

Evolution of 41421736 to its Cube Root, makes 346.

Thus:

$$\begin{array}{l} \text{Thus:} \\ \hline \begin{array}{r} \text{A} \quad \text{B} \\ \text{41421736} \begin{pmatrix} a & b \\ 300 + 40 + 6. \end{pmatrix} \\ a^3 = 27000000 \\ \hline 3a^2 + 3a = 270900 \quad 14421736 \text{ Resolvend.} \\ \hline 10800000 = 3a^2b. \\ 1440000 = 3ab^2. \\ 64000 = b^3. \\ \hline 12304000 \text{ Sum.} \\ \hline 3A^2 + 3A = 347820 \quad 2117736 \text{ Resolvend.} \\ \hline 2080800 = 3A^2B. \\ 36720 = 3AB^2. \\ 216 = B^3. \\ \hline 2117736 \text{ Sum.} \\ \hline 000000 \end{array} \end{array}$$

There remain yet two things to be demonstrated, which are deliver'd in *Schol.* 1 and 2. viz.

1. The Remainder can never exceed the Sum of these two Numbers, viz. triple all the Figures of the Root already found (taken as one Number) and triple the Square of the same; the Reason of which you have plainly in *Cor. Lem.* 4.

2. If the Divisor $3a^2 + 3a$ is contained oftner than 9 times in the Resolvend, then if it's so after the second Figure is found, 9 is the Figure sought; for in this Case the Resolvend may be thus represented, $3a^2 \times 10 + 3a \times 10 + R = 3a^2 \times 9 + 3a^2 + 3a \times 10 + R$, and the Sum upon which the Limitation depends, being $3a^2b + 3ab^2 + b^3$, if b is 9, then this Sum is $3a^2 \times 9 + 3a \times 81 + 729$: Compare this with the Resolvend, they have this Part in common, viz. $3a^2 \times 9$: Set this aside, and compare the Remainders in both, viz. $3a^2 + 3a \times 10 + R$, and $3a \times 81 + 729$; this last is less than the former; for, after two Figures of the Root are found, a consists of three Figures, in its compleat Value, and so must be at least 100: Therefore $3a \times 81$ is less than $3a^2$, and 729 is less than $3a \times 10$, which is at least $300 \times 10 = 3000$. Hence it is plain, that the Resolvend is greater than the Number to be compared with it in the Limitation of the Quote; and the greater that A is,

\bar{a} is, as it will be always greater at every Step after the second, so much will the Resolvend exceed that other Number.

Again; In the second Step, the Quote is certainly 9, if the first Figure found is either 8 or 9; *i. e.* if $A = 80$ or 90 ; which you'll find by comparing as before $3A^2 + 3A \times 10 + R$ with $3a \times 81 + 729$; for putting $a = 80$ or 90 , you'll find $3A \times 81 + 729$ less than $3A^2 + 3A \times 10$: But when A is supposed 70, or 60, &c. it will be greater, and therefore the Quote must be less than 9, unless the Number R , which belongs to the Resolvend, is greater than the Excess of $3A \times 81 + 729$ above $3A^2 + 3A \times 10$, as in some Cases it will, and in some it will not.

S C H O L I U M, concerning a different Method of Practice in the Extraction of
a Cube Root.

The preceding Rule is nearly according to the most common Method, that it might be accommodated to the Principles from which the Reason and Demonstration of it might be most easily deduced: But there is another Method, differing a little in one of the principal Steps, which is this:

Having pointed the given Number, and found the first Figure of the Root; then in all the succeeding Steps form the Divisor as before, and find the Quote under this Limitation, *viz.* That being added to the Divisor, and the Sum multiplied by the same, the Product shall be less than the Resolvend; which is so far like what we do for the Square Root: But, again; the Remainder must not exceed the Product of these Numbers, *viz.* the Sum of the Quote and the second Member of the Divisor multiplied into the Difference betwixt the Quote and its Square; *i. e.* add together these two Products, and their Sum must not exceed the Resolvend, and what remains here belongs to the next Resolvend.

You may also form your Divisor thus; Take the Figures already found, and to them prefix 1 (or take them with 0 prefix'd, and then add 1, which will fall in the Place of the 0); multiply this Sum by triple the Number to which the 1 was added: The Product is the Divisor. See this *Example* wrought after this Manner.

Divisor.	$\begin{array}{r} 614125 \quad (a.b \\ 512 = a^3 \end{array}$
$3a^2 + 3a = 19440$ $(= 3a \times a + 1 =$ $3 \times 80 \times 81)$	$102125. \text{ 2d Resolvend.}$ <hr/> $97225 = 3a^2 + 3a + b \times b$ $4900 = 3a + b \times b^2 - b$ <hr/> $102125. \text{ Sum.}$ <hr/> 000000 <hr/>

The Letters and Operations shew the Application of this Method; and what is to be demonstrated is only this, that the Number compared to the Resolvend is equal to $3a^2b + 3ab^2 + b^3$, which is the Number compared in the former Rule; and the Truth of this you'll find by performing the Operation of these two Products, and adding them thus, $3a^2 + 3a + b \times b = 3a^2b + 3ab + b^2$, then $3a + b \times b^2 - b = 3ab^2 + b^3 - 3ab - b^2$: which added to the former makes $3a^2b + 3ab^2 + b^3$.

What I have further to observe is, That this Method will in many Cases be of Advantage, by helping us to discover more easily that some Figures are too great for the Quote, without the Trouble of making out the total Number, which is here to be compared with the Resolvend: For if the first Part of it (*viz.* the Product of the Quote by the Sum of the Divisor and Quote, or $3a^2 + 3a + b \times b$) is equal to the Resolvend, or greater,

greater, that Figure is certainly too big to answer the Rule. But tho' that first Part is less than the Resolvend, we cannot conclude that we have the true Figure, till we add the other Part also, and find that the Sum is not greater than the Resolvend.

Observe also, That if the Product of the Quote b , and first Member of the Divisor, viz. $3a^2$, is equal to the Resolvend or greater, then certainly that Quote is too big, and so we might have the same kind of Advantage by the common Method; yet the Product of the Quote into the Sum of the Quote and Divisor, being always a greater Number than the Product of the Quote and first Member of the Divisor, the last Method will discover some Figures to be too great, which would not appear so without Trial by the other Method.

Of the Proof of the Cube Root.

Involve the Root found to the Cube, and compare it with the given Cube, or the Difference betwixt the given Number and the Remainder of the Extraction.

Or, By casting out 9's thus: Cast the 9's out of the given Number, if there is no Remainder in the Extraction; or out of the Difference of that Number and the Remainder of the Extraction: Then cast the 9's out of the Root found, and square the Excess, out of which cast the 9's, and multiply this Excess by the preceding, and out of this Product cast the 9's; the Excess or Defect of 9 must be equal to that found in the given Number.

Example: The Cube Root of 2744 is 14; thus proved: The Excess of 9's in 2744 is 8. in 14 it is 5; then $5 \times 5 = 25$, in which the Excess of 9's is 7, which, multiplied by the preceding Excess 5, the Product is 35, in which the Excess of 9's is 8.

$14 \times 14 \times 14 = 2744$ | The Reason of this Practice is also obvious from what is shewn in Multiplication: For taking 14×14 as one Factor, and 14 as another, we first cast out 9's out of 14×14 , and then out of 14; and, multiplying these two Excesses together, we compare the Excess of 9's in the Product with that in 2744, which is $= 14 \times 14 \times 14$.

Problem III. *To Extract the Root of any Power above the Cube.*

General RULE.

Whatever Root is proposed to be extracted, as in general the n Root, distribute the given Number into Periods, taking as many Figures to each Period as the Number of Units in the Index n ; then make a Table of the similar Powers (*i. e.* the n Powers) of all the Digits, as far at least till you find one which is equal to, or exceeds the first Period on the Left of the given Number, taken by itself; the Root of that Power is the first Figure on the Left of the Root sought, which call A ; then subtract A^n from the said first Period, and to the Remainder prefix the next Period for a Resolvend; and to find the next Figure of the Root, form a Divisor thus; take a Binomial $A + B$, and involve it to the n Power, as has been explained; your Divisor is the Sum of all the Products of the several Powers of A , except the highest A^n , multiplied by the proper Coefficients of the Terms in which they stand in the Power $A + B^n$: Thus, for the 4th Root the Divisor is $4A^3 + 6A^2 + 4A$; for the 5th Root it is $5A^4 + 10A^3 + 10A^2 + 5A$, as you'll find from the *Table of Binomial Powers and Coefficients*. And universally, the Divisor will be $nA^{n-1} + a \times A^{n-2} + b \times A^{n-3} + \&c. + n \times A$, where I have simply expressed the Coefficients by single Letters, which you must understand as representing the true Coefficients. Also remember, that the first Figure of the Root found, which A represents, must be multiplied

multiplied by 10, or 0 prefix'd to it, because that is its true Value with respect to the next Figure to be found; and in this Value you are to use it in forming the Divisor; then find the Quote B, (which can never exceed 9) limited so that the several Members of the Divisor being multiplied by the several Powers of B, which you find multiplied into them in the several Terms of the Binomial Power $\overline{A+B^n}$; and B^n added to the Sum of all these Products, the total shall not exceed the Resolvend. Thus in the 4th Power the Number to be compared with the Resolvend is this, $4 A^3 B + 6 A^2 B^2 + 4 A B^3 + B^4$; in the 5th Power it is $5 A^4 B + 10 A^3 B^2 + 10 A^2 B^3 + 5 A B^4 + B^5$. Universally, it is $n A^{n-1} \times B + 2 A^{n-2} \times B^2 + 6 A^{n-3} \times B^3 + \&c. + n A B^{n-1} + B^n$, which is the whole Binomial Power except the Term A^n .

In the next place take the two Figures found, and prefixing 0, call this Number A, and form a new Divisor as before, of the several Powers of this new Number A, multiplied by their Coefficients in $\overline{A+B^n}$, and by this find the 3d Figure of the Root, which call again B, under the same Limitation as before; and so proceed to the End.

S C H O L I U M S.

1. If you begin your Guess for the Quote (*i.e.* for any Figure of the Root after the first) at the greatest number of times (not exceeding 9) that the Divisor is contained in the Resolvend, then the Limitation of the Rule for the Number to be compared with the Resolvend is sufficient to determine when you have the true Figure. But if you chuse at a Guess, then you are to mind this Mark of a Figure too little, *viz.* That if you take all the Root found, taking in the Figure now put in the Root, and call it A; then take the Sum of the Products of the several Powers of it (except A^n) which belong to the new Divisor; the Remainder must not exceed this, else the Figure last found is too little.

2. If there is a Remainder after all the Periods are employed, the given Number is not a Power of that Order, and the Root found is only that of the greatest Power contained in it.

Roots. 5th Powers.

1	:	1
2	:	32
3	:	243
4	:	1024

I shall illustrate this Rule as far as is necessary by an Example. Suppose the 5th Root of this Number 74560898 is required. Having pointed it, the last Period is 745; and raising the 5th Powers of the Numbers from 1 to 4, whose 5th Power is the first which exceeds 745, I find 243 the greatest 5th Power contained in 745; and the 5th Root of this being 3, I put 3 as the last Figure of the Root sought.

$$A^5 = \begin{array}{r} 74560898 \\ 243 \\ \hline 4329150 \end{array} \begin{array}{l} \text{AB} \\ 37 \end{array}$$

50260898. 2d Resolvend.

$$28350000 = 5 A^4 \times B \quad \left(\begin{array}{l} A = 30 \\ B = 7 \end{array} \right)$$

$$13230000 = 10 A^3 \times B^2$$

$$3087000 = 10 A^2 \times B^3$$

$$360150 = 5 A \times B^4$$

$$16807 = B^5$$

$$\hline 45043957. \text{ Sum.}$$

$$\hline 5216941. \text{ Remainder.}$$

Then raising $\overline{A+B^5}$, it is $A^5 + 5 A^4 B + 10 A^3 B^2 + 10 A^2 B^3 + 5 A B^4 + B^5$; and taking $A = 30$, the Divisor is $5 A^4 + 10 A^3 + 10 A^2 + 5 A = 4329150$, which is found in the Resolvend, under the Limitation of the Rule 7 times; the Remainder being 5216941.

The Divisor being formed thus:

$$\begin{array}{r} 5 A^4 = 4050000 \\ 10 A^3 = 270000 \\ 10 A^2 = 9000 \\ 5 A = 150 \\ \hline 4329150 \end{array}$$

DEMONSTR. The Demonstration of this general Rule depends upon the same kind of Principles as those for the Square and Cube: And whoever understands these thoroughly will be able to extend them to this universal Rule with great Ease: For if we put n in the Place of 2 or 3 in the preceding *Lemma's*, they will become universal for all Cases.

SCHOLIUM. What a tedious thing it is to form the Divisors, and the Numbers to be compared with the Resolvend in high Powers, and indeed in all above the Cube, it's easy to perceive. All that can be said in favour of this general Rule is only this, That it is exceedingly preferable to our being left to a pure blind Guess, with no other Help than raising the Power of the Root guessed, and comparing it with the proposed Number. Yet the great Labour of this Rule has excited the Mathematicians to the Invention of other Methods; the explaining of which comes not within the Limits I have prescribed my self in this Work, except that Method which is by the help of *Logarithms*, as you'll find afterwards explained. In the mean time *observe*, that as Square and Cube Roots are the things only useful in the common Affairs of Life, so the Rules for them are tolerably easy, especially the Square. But there is also

Another General RULE for Compound Roots (i. e. whose Index is the Product of two or more Numbers).

Take any two or more Indexes whose Product is the given *Index*, and extract out of the given Number a Root answering to any of these lesser Indexes; and then out of this Root extract a Root answering to another of these lesser Indexes, and so on, till you go thro' them all: The last Root found is the Root sought.

Example 1. To find the 4th Root of 625, I find the Square Root 25; then the Square Root of this, which is 5, is the Root sought.

Example 2. To find the 6th Root of 4096: It is 4; which I find thus: $6 = 2 \times 3$, therefore I find the Square Root of 4096, which is 64, and then the 3d Root of 64 is 4.

Demonstr. The Reason of this Rule is obvious, being only the Reverse of what's done and demonstrated for involving a Number to a compound Power; or you have the Reason of it in *Theor. IX. §. 1.* where it's shewn that $A^{\frac{1}{nm}} = A^{\frac{1}{n} \cdot \frac{1}{m}}$.

Observe, It's best to begin with the Root of the lowest Index.

Also, If the given Number is not a Power of the Order you first try, neither can it be a Power of the Order proposed; and to find the Root of the greatest like Power contained in it, other Methods do better.

Of the Proof of all Roots of Integers universally.

It is done either by the opposite Involution, or by casting out the 9's, thus:

Cast the 9's out of the given Number, or the Difference of it and the Remainder of the Extraction, and mark the Excess: Then cast the 9's out of the Root (and take the Excess, or the Root itself if less than 9); multiply it by itself, and cast out the 9's from the Product; then multiply the Excess by the Excess in the Root, and cast the 9's out of the Product; this last Excess multiply by the Excess in the Root, and cast the 9's out of the Product, and go on so till the Excess of 9's in the Root is employ'd as a Multiplier, as oft

oft as the Index of the Power expreffes: The laft Excefs muft be equal to that in the given Number.

§. II. PART II.

Probl. 4. Of the Extraction of the Roots of Fractions.

A Fractional Power is to be confidered in two different Views: 1. As being an immediate Power, *i. e.* the immediate Effect of the continual Multiplication of fome Fraction into itfelf, as $\frac{4}{9} = \frac{2}{3} \times \frac{2}{3}$; and $\frac{8}{27} = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}$. Or, 2. As being only equivalent to fome immediate Power, but not itfelf fuch a one, as $\frac{8}{18} = \frac{4}{9}$.

Now if a Fraction is immediately a Power, it's manifef from the Definitions, that if we extract the Root propofed from the Numerator and Denominator feparately, thefe are the Numerator and Denominator of the fractional Root fought. *Example*: The Square Root of $\frac{64}{81}$ is $\frac{8}{9}$; for $8 = 64^{\frac{1}{2}}$, and $9 = 81^{\frac{1}{2}}$. But if the given Fraction is only equivalent to fome immediate Power, the Root (*viz.* of that Power; which is alfo in another Senfe, the Root of the given Fraction) cannot be difcovered by this Method; for the Numerator and Denominator have not both in this Cafe, and perhaps neither of them has a perfect Root; and fo we cannot determine by this Method, whether the Root fought is rational or furd: yet by other Methods we can difcover this, and find the Root where there is one. For which take this

General RULE.

Reduce the given Fraction to its loweft Terms, and then extract the propofed Root from Numerator and Denominator feparately; and thefe Roots are the Numerator and Denominator of the Fractional Root fought; which is alfo in its loweft Terms. But if both Numerator and Denominator have not fuch a perfect Root, the given Number is not a Power of the Order propofed, either immediately or equivalently.

Example 1. To find the Square Root of $\frac{27}{75}$, I find its leaft Terms $\frac{9}{25}$, whole immediate Root is $\frac{3}{5}$.

Example 2. To find the Square Root of $\frac{24}{69}$, I find its leaft Terms $\frac{8}{23}$. But neither 8 nor 23 are Squares, and therefore $\frac{24}{69}$ is not a Square in any Senfe.

DEMON. 1. If the loweft (or any) Terms of the given Fraction are Powers of the Order propofed, it's plain that their Roots make a Fraction, which is the Root of the given Fraction; by the Definition. And,

2. If the leaft Terms of a Fraction are not Powers of the given Order, no Terms of it are fo; or the given Fraction is not a Power in any Senfe. For let $\frac{A}{B}$ be a Fraction in its leaft Terms, and fuppofe $\frac{M^n}{N^n} = \frac{A}{B}$ (*i. e.* fome other equivalent Terms of the Fraction to be an immediate Power.) Then becaufe $\frac{A}{B}$ is in leaft Terms, $\frac{M^n}{N^n}$ is not fo, becaufe it confifts of different Terms by Suppofition. Confequently $\frac{M}{N}$, its n Root, is not

in its least Terms, (by *Lem.* preceding the *Theor.* Chap. 1.) Take $\frac{r}{s}$ in its least Terms, and $= \frac{M}{N}$; then is $\frac{r^n}{s^n}$ in its least Terms, (by the same *Lem.*) And since $\frac{r}{s} = \frac{M}{N}$, therefore $\frac{r^n}{s^n} = \frac{M^n}{N^n} = \frac{A}{B}$; wherefore $\frac{r^n}{s^n}$ and $\frac{A}{B}$ are both in the least Terms, which is absurd; or $\frac{r^n}{s^n} = \frac{A}{B}$ are the same Terms, which is also contrary to Supposition.

SCHOLIUMS.

1. A Fraction made of the greatest Integral Root of the Numerator and Denominator may in one Sense be called the Root of the greatest Fractional Power contained in the given Fraction; which Root will, in some Cases, be a deficient, and in some an excessive Root, *i. e.* whose Power wants of, or exceeds the given Fraction. *Example:* $\frac{5}{13}$, the greatest Square Fraction contained in it, in this Sense, is $\frac{4}{9}$, whose Root is $\frac{2}{3}$; which is an excessive Root to $\frac{5}{13}$, because $\frac{4}{9}$ is a greater Fraction than $\frac{5}{13}$. But in $\frac{8}{15}$ the greatest Square is $\frac{1}{9}$, which is less than $\frac{8}{15}$; therefore its Root $\frac{1}{3}$ is a deficient Root to $\frac{8}{15}$.

Again: In another Sense, *i. e.* if we ask what is the greatest Fraction which is an immediate Power, and is less than a given Fraction which is not a Power in any Sense, then there is no such thing as a greatest; the Reason of which you'll find afterwards. (See *Corol. Prob. 5.*)

2. The preceding General Rule requires two Extractions, *viz.* both from the Numerator and Denominator; but I shall give you other particular Rules, whereby the Root is found by one Extraction; and such as are accommodated to the Methods of Approximation, afterwards explain'd.

Particular RULES for the Roots of Fractions.

1. For the Square Root.

Multiply the Numerator and Denominator together, and extract the Square Root of the Product; which is always a compleat Square, if the given Fraction is so in any Sense. Make this Root the Numerator to the given Denominator, and this Fraction is the Root sought; or set the given Numerator fractionally over the Root found; and this also is the Root sought, tho' neither of them is in the least Terms. But if the Product is not a compleat Square, neither is the given Fraction: And having found the Root of the greatest Integral Square contained in it, use that as directed; and you shall have a Root wanting of a just Root to the given Fraction, if the Root extracted is made Numerator; but exceeding, if it's made Denominator.

Example 1. To find the Square Root of $\frac{4}{9}$, I take $4 \times 9 = 36$, whose Root is 6; and so the Root sought is $\frac{6}{9} = \frac{2}{3}$, or $\frac{4}{6} = \frac{2}{3}$; for $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$.

Example 2. To find the Square Root of $\frac{12}{147}$, I take $12 \times 147 = 1764$, whose Root is 42; and so $\frac{42}{147} = \frac{12}{42} = \frac{2}{7}$ is the Root sought; for $\frac{2}{7} \times \frac{2}{7} = \frac{4}{49} = \frac{12}{147}$.

Example 3. For the Square Root of $\frac{7}{12}$, I take $7 \times 12 = 84$, which is not a Square; therefore $\frac{7}{12}$ is not so: But the greatest Integral Root in 84 is 9, therefore $\frac{9}{12} = \frac{3}{4}$ is a Root wanting of a true Root to $\frac{7}{12}$; for $\frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$, which is less than $\frac{7}{12}$, because 9×12 is less than 7×16 . And $\frac{7}{9}$ is an excessive Root; for $\frac{7}{9} \times \frac{7}{9} = \frac{49}{81}$ greater than $\frac{7}{12}$, because 49×12 is greater than 7×81 .

SCHOLIUM. Tho' the Fraction given be in its least Terms, yet the Root found by this Method will not be in its least Terms in every Case where the given Fraction is not a perfect Power, as the preceding *Exam. 3.* shews. And if it is a perfect Power, the Root found will never be in its least Terms, as is manifest; because the least Terms are the Root of the Numerator and Denominator of the least Terms of the given Fraction.

DEMON. 1. The given Fraction being $\frac{A}{B}$, multiply both Terms by B, or by A; and then $\frac{A}{B} = \frac{AB}{BB} = \frac{AA}{AB}$. Suppose AB is a compleat Square, whose Root is a , so that $AB = aa$; then are all these Expressions equal, viz. $\frac{aa}{BB} = \frac{AB}{BB} = \frac{A}{B} = \frac{AA}{AB} = \frac{AA}{aa}$; consequently $\frac{a}{B}$ the Square Root of $\frac{aa}{BB}$, and $\frac{A}{a}$ the Square Root of $\frac{AA}{aa}$ are each a true Root to $\frac{A}{B}$.

2. If AB is not a perfect Square, then neither is $\frac{A}{B}$ in any Sense. For suppose $\frac{M^2}{N^2} = \frac{A}{B} = \frac{AB}{BB}$, then, by equal Multiplication, according to the Nature of Fractions, it is $\frac{M^2 \times B^2}{N^2} = AB$. But $M^2 \times B^2 = \overline{MB}^2$, therefore $\frac{M^2 \times B^2}{N^2} = \frac{\overline{MB}^2}{N^2}$; which is plainly an immediate Square, whose Root is $\frac{MB}{N}$, and is therefore a true Root to AB, which is absurd; for AB is supposed not to have a true Square Root in Integers, and consequently has no such Root true, (*Theor. 19. Chap. 1.*) therefore $\frac{MB}{N}$ is not the Square Root of AB; Nor is $\frac{M^2 \times B^2}{N^2}$ equal to AB; nor $\frac{M^2}{N^2}$ equal to $\frac{AB}{BB} (= \frac{A}{B})$ as was supposed, i. e. no immediate Square $\frac{M^2}{N^2}$ can be equal to $\frac{A}{B}$, or $\frac{A}{B}$ is not a Square in any Sense.

3. Suppose \bar{a} the greatest Integral Root of AB, so that aa is less than AB; then is $\frac{aa}{BB}$ less than $\frac{AB}{BB} (= \frac{A}{B})$ i. e. $\frac{a}{B}$ is a deficient Root to $\frac{A}{B}$. Also since aa is less than AB, therefore $\frac{AA}{aa}$ is greater than $\frac{AA}{AB} (= \frac{AB}{BB} = \frac{A}{B})$ i. e. $\frac{A}{a}$ is an excessive Root to $\frac{A}{B}$.

2. For the Cube Root.

Multiply the Numerator by the Square of the Denominator; the Product will be a compleat Cube, if the given Fraction is so in any Sense: the Cube Root of this Product set fractionally over the Denominator of the given Fraction, is the fractional Root sought.

sought. Or *thus*: Multiply the Denominator by the Square of the Numerator, and over the Cube Root of the Product set the given Numerator fractionally; and this also is the Root sought. And *observe*, that if the given Fraction is a proper one, the last is the best Method; but if it is improper, take the first Method. But if the Product mention'd is not a compleat Cube, neither is the given Fraction. And if we take the greatest Integral Root of that Product, and use it as directed, we have a deficient or excessive Root, according as we chuse the first or second Method.

Example 1. To find the Cube Root of $\frac{8}{27}$, take $27 \times 27 = 729$, then $729 \times 8 = 5832$, whose Cube Root is 18; then is $\frac{18}{27} = \frac{2}{3}$ the Root sought. Or thus; $8 \times 8 = 64$, then $64 \times 27 = 1728$, whose Cube Root is 12; therefore $\frac{8}{12} = \frac{2}{3}$ is also the Root sought.

Example 2. For $\frac{15}{54}$, take $54 \times 54 = 2916$, then $2916 \times 15 = 43740$, which has not a Cube Root, and so $\frac{15}{54}$ is not a Cube; but the greatest integral Cube Root of 43740 being 35, therefore $\frac{35}{54}$ is a deficient Root to $\frac{15}{54}$; or if we take $15 \times 15 = 225$, then $225 \times 54 = 12150$, whose nearest Cube Root is 22, and $\frac{15}{22}$ is an Excessive Root.

DEMON. 1. For the first Method, multiply each Term of the Fraction $\frac{A}{B}$ by B^2 , and it is $\frac{A B^2}{B^3} = \frac{A}{B}$; and if $A B^2$ is a Cube, let its Root be m , then $\frac{m}{B}$ is the Cube Root of $\frac{A B^2}{B^3} = \frac{A}{B}$. Again; if $A B^2$ is not a Cube, neither is $\frac{A}{B}$; for if we suppose $\frac{m^3}{n^3} = \frac{A B^2}{B^3}$, then is $A B^2 = \frac{m^3 \times B^3}{n^3} = \frac{m^3 B^3}{n^3}$, whose Cube Root is $\frac{m B}{n}$; i. e. $A B^2$ is a Cube, which is contrary to Supposition, if $\frac{m B}{n}$ is Integer; and if it's not Integer, it cannot be the Cube Root of $A B^2$, which has no Cube Root in Integers (*Theor. XIX. Ch. 1.*). But if we suppose m the greatest integral Cube Root contained in $A B^2$, so that m^3 is less than $A B^2$, it's plain that $\frac{m^3}{B^3}$ is less than $\frac{A B^2}{B^3} = \frac{A}{B}$, or $\frac{m}{B}$ is a deficient Root.

2. For the second Method, multiply each Term of $\frac{A}{B}$ by A^2 , and it is $\frac{A^3}{A^2 B} = \frac{A}{B}$, so that $A^2 B$ being a true Cube, whose Root is n , then $\frac{A}{n}$ is the Root of $\frac{A^3}{A^2 B} = \frac{A}{B}$; but if $A^2 B$ is not a Cube, neither is $\frac{A}{B}$; for suppose $\frac{m^3}{n^3} = \frac{A^3}{A^2 B}$, then is $\frac{m^3}{n^3} \times A^2 B = A^3$, and $A^2 B = A^3 \div \frac{m^3}{n^3} = \frac{A^3 \times n^3}{m^3} = \frac{A n^3}{m^3}$, whose Cube Root is $\frac{A n}{m}$; i. e. $A^2 B$ is a Cube, contrary to Supposition: But if we take m the greatest integral Cube Root contained in $A^2 B$, so that m^3 is less than $A^2 B$, then is $\frac{A^3}{m^3}$ greater than $\frac{A^3}{A^2 B}$, and consequently $\frac{A}{m}$ is an excessive Root to $\frac{A}{B}$.

SCHOLIUM. Both these, and the Extraction of all higher Roots, may be comprehended in one general Rule, thus:

General RULE for all the Roots of Fractions, after the Manner of the preceding Particular Rules.

Raise the Denominator of the given Fraction to a Power, whose Index is 1 less than that of the Root to be extracted, and multiply this Power by the given Numerator: Extract the proposed Root of this Product (which is a compleat Power, if the given Fraction is so), and set it fractionally over the given Denominator, and this makes the Root sought. But if that Product has not such a Root, neither has the given Fraction; and taking the greatest integral Root contained in that Product, it makes, with the given Denominator, a deficient Root to the given Fraction.

Or also thus: Raise the Numerator to the Power directed, and by that multiply the Denominator; extract the proposed Root of the Product if it has one, or take the greatest integral Root contained in it; over this Root set the given Numerator, and it makes an excessive Root to the given Fraction.

Example. To find the 4th Root of $\frac{A}{B}$, I multiply B^3 by A , and set the 4th Root of the Product AB^3 , or the greatest integral 4th Power contained in it, over B , and it makes the Root sought, or a deficient one; or also over the 4th Root of A^3B set A , it makes the Root sought, or an excessive one.

DEMON. 1. For the n Root of $\frac{A}{B}$: If AB^{n-1} is a true Power of the Order n , let its Root be m , then it's plain that $\frac{m}{B}$ is the n Root of $\frac{AB^{n-1}}{B^n} = \frac{AB^{n-1}}{B \times B^{n-1}} = \frac{A}{B}$. And if m is only the Root of the greatest integral Power contained in AB^{n-1} , it self not being one, then is $\frac{m}{B}$ plainly a deficient Root to $\frac{A}{B}$, which in this Case has no true Root; for if we suppose $\frac{m^n}{B^n} = \frac{AB^{n-1}}{B^n} (= \frac{A}{B})$, then is $AB^{n-1} = \frac{m \times B^n}{n^n} = \frac{mB^n}{n^n}$, whose n Root is $\frac{mB}{n}$; i. e. AB^{n-1} is a Power of the Order n , contrary to Supposition.

2. For the 2d Method; $\frac{A^n}{BA^{n-1}} = \frac{A}{B}$, and if the n Root of BA^{n-1} is n , then is $\frac{A}{n}$ the Root sought: But if n is only the Root of the greatest integral Power contain'd in BA^{n-1} , it self not being one, then is $\frac{A}{n}$ an excessive Root to $\frac{A}{B}$, which in this Case has no true Root; for if $\frac{m^n}{n^n} = \frac{A^n}{BA^{n-1}}$, then is $BA^{n-1} = \frac{A^n \times n^n}{m^n}$, whose n Root is $\frac{A \times n}{m}$; i. e. BA^{n-1} is a Power of the Order n , contrary to Supposition.

SCHOLIUM. If the Denominator of the Root is a Compound Number; i. e. the Product of two or more Integers, the Extraction may be made by several more simple Extractions, in the manner already explained, which needs not to be further insisted on.

§. II. PART III.

Problem 5. Of the Approximation of Roots.

DEFINITION.

WE have already observed, that tho' a Number has no determinate Root, yet it has what we may call an indeterminate one (ordinarily called a *Surd* Root); *i.e.* there is a certain Series of Numbers decreasing, which can be carried on by a certain Law or Order *in infinitum*, whose Sum taken from the beginning is a Root whose Power approaches nearer and nearer to the given Number, as the Series goes on; and tho' it is never equal to it precisely, it may be brought within any assignable Difference: The *Invention*, or carrying on of this Series is what we here call the *Approximation* of the Root; and if we take the Series of the Sums invented, it may be called the Series of *Approaching* Roots. Observe also, that they may be found approaching yet either still less or still greater than true Roots.

I. For Roots of Integers.

RULE. Whatever Root is proposed, after the Root of the greatest integral Power contained in the given Number is found, by the preceding Rules; To the Remainder prefix a Period of 0's according to the Index; thus 00 for a Square Root, 000 for a Cube, 0000 for a 4th Power, and so on: Then form a Divisor, and find a new Figure in the Root the same way as in the preceding Steps of the Work: To every succeeding Remainder prefix a Period of 0's, and find a new Figure of the Root, and this Work will go on for ever, because there will always be a Remainder. The Figures thus found are all Decimal Places in the Root, the decimal Point being placed immediately after the integral Part, and before these new Figures. And thus we have a Mixt Number for the Root; which is still nearer and nearer to the true Root of the given Number, the further the Operation is carried on, but is still deficient, because there is still a Remainder. Again; Observe, that if to the last Figure found in the Root you add 1, the Sum will make an excessive Root; and thus you may have a Series of Roots nearer and nearer, but still excessive.

The following Example of a Square Root will sufficiently illustrate this Practice.

Operation.

387 (19.672, &c.
 1
 —
 287
 261
 —
 2600
 2316
 —
 28400
 27489
 —
 91100
 78684
 —
 12416
 &c.

To find the Square Root of 387. The Root of the greatest integral Square contained in it is 19. Then by one Period of 0's the Root becomes 19.6; by a 2d it is 19.67; by a 3d it is 19.672; and may be carried further at pleasure; and each of these Roots are deficient; *i.e.* their Figures are less than 387; but the Difference is still less and less: and what I called the Series of Numbers decreasing, whose Sums make the Series of approaching Roots, tho' still defective, are these 19, .6, .07, .002, &c. and the Series of their Sums, which make the approaching Root, is 19, 19.6, 19.67, 19.672, &c. And lastly, by adding 1 to each of these, we have a Series of approaching Roots, but still excessive, tho' the Differences grow still less. Thus; 20, 19.7, 19.68, 19.673, &c.

DEMON. I. If any compleat integral Power of any Order is multiplied into a Number which is not a Power of that Order, the Product is not a Power of that Order; or has not a perfect Root of that Order. Thus; If A is not a Power of the Order n , neither is $A \times B^n$, as has been demonstrated in *Theor. II. Coroll. 4. Chap. I.*

2. If

2. If the Root of the greatest integral Power contained in AB^n is divided by B , which is the n Root of the Multiplier B^n , the Quotient is less than a true Root to the given Number. For suppose r to be the n Root of the greatest integral Power of the Order n contained in AB^n , and it's plain that r^n is less than AB^n , therefore take their like aliquot Parts, and $\frac{r^n}{B^n}$ is less than $\frac{AB^n}{B^n}$, or A ; *i.e.* $\frac{r}{B}$, the n Root of $\frac{r^n}{B^n}$, is less than a true Root to A . Again; If to r , the greatest integral Root of AB^n , we add 1, and call the Sum s , then s^n is greater than AB^n ; and consequently $\frac{s^n}{B^n}$ greater than $\frac{AB^n}{B^n}$, or A ; *i.e.* $\frac{s}{B}$ is an excessive Root to A .

From these two Articles we shall easily demonstrate the Rule of *Approximation*, thus:

3. The greatest integral Root, or Root of the greatest integral Power contained in the given Number being found, what remains to be proved is this only, That the Extraction will go on in this manner without end; *i.e.* that there will always be a Remainder, and so a new decimal Fraction will at every Step be added to the preceding Root, making the whole greater and greater; yet so that the Mixt Root will still be deficient, or its Power still less than the given Number, tho' still nearer *in infinitum*. To shew this Truth, consider, that by prefixing Periods of o's to any Number, we do really multiply it by a Number consisting of 1 with as many o's as are thus prefix'd; but it's the same thing to multiply the given Number (whose Root we seek) by prefixing o's, and then bringing them down to the Remainders, or prefixing them only to the Remainders; for either way we find the Root of the Product (or the greatest integral Root contained in it). Thus, for a Square Root one Period 00 multiplies the given Number by 100, two Periods multiplies by 10000, &c. For a Cube Root one Period 000 multiplies by 1000, and two Periods multiplies by 1000000, &c. and so of other Powers. But these Multipliers are evidently true and compleat Powers of their several Orders, whose Roots are 1, with as many o's as we have used Periods of o's; therefore, by the first Article, however far the Extraction is carried by Periods of o's thus prefix'd to the Remainders, *i.e.* however great the Power is by which we have thus multiplied the given Number, there will always be a Remainder, because the given Number not being a true Power, tho' the Multiplier is, yet the Product is not. Again; By putting all the Figures found by means of these Periods of o's, in decimal Places, we do evidently divide the Root of the Product, (*i.e.* the Root of the greatest integral Power contain'd in it) by the Root of the Number multiplied into the given Number: Because for every Period annex'd we have one Place in Decimals; which is plainly dividing the Root found, considered all as a whole Number, by 10, or 100, &c. according to the Number of Periods of o's employ'd. Therefore, by the 2d Article, this Mixt Root will always be less than a just Root to the given Number, tho' still approaching nearer, which demonstrates the Rule as to the Series of deficient Roots: and as to the excessive Roots, it's evident that adding an Unit to the last Place of the Root already found, is adding 1 to the Root of the greatest integral Power contained in the given Number, or to its Product by the Power which multiplies it: Therefore, by the 2d Article, the Root becomes excessive. Or it's found by this Consideration, That 1 in any Place of a Number either integral or decimal, is of more value than all the rest of the Number standing on the Right of that Place, however many Figures there be.

SCHOLIUMS.

1. The Proof of this Operation is made the same way as has been already explain'd, *viz.* either by raising the Root found to its Power, and adding the Remainder; or by casting out of the 9's.

As to the former Method, *Observe*, That we need to take no notice of the Root's being a Mixt Number, but take it all as a whole Number, and the Remainder so also; and then

then the Sum of the Power and Remainder must have as many Periods of o's on the Right as were used in the Operation; because when the Root and Remainder are taken for Integers, so many Periods of o's belong to the supposed Power or Number, whereof that Root is the greatest integral Root: But if we take the Root as it really is, a Mixt Number, then the Remainder is a decimal Fraction, whose Denominator is 1, with as many o's as were added to all the Remainders in the Operation, and in this value it is to be added to the Power of the Mixt Root: Thus in the preceding Example, the Root found is 19.672, whose Square is 386.987584, and the Remainder is, in its true value, .012416; for two Periods, or 6 o's, were employed in the Operation; and the Sum of 386.987584, + .012416 is = 387.000000 (= 387), which is the same as if the Quote and Remainder had been taken for Integers, and the given Number had been 387000000.

As to the Method by casting out 9's; When we subtract the Remainder from the given Number, we may take it either in its real Value, or as a whole Number, and then we must take the given Number, with as many o's after it as were used in the Operation. For it is the same to the present Purpose, to take .012416 from 387, or 12416 from 387000000, the Remainder in both Cases being the same Figures, viz. 386.987584, or 386987584.

2. If we point the true Value of the Remainder at every Step of the Approximation, this will shew gradually how much the proposed Power of the Root found wants of the given Number; and as the Root, so consequently its Power continually increases; therefore these Remainders will continually diminish; so that by observing this, we can carry on the Work till that Difference or Remainder be as little as we please, or less than any assigned Difference.

But if instead of this, it should be required to extract the Root so near to a true and perfect Root to the given Number that it shall want less than an assigned Difference, i. e. so that this Difference added to the Root found, the Power of the Sum shall exceed the given Number, it's done thus; Suppose any Fraction $\frac{a}{r}$ to be the given Difference, within which the Root is to be brought; then extract the Root to a Number of decimal Places equal to the Number of Figures in r , and you have done; for $\frac{1}{r}$ is less than $\frac{a}{r}$, if a is greater than 1; and a decimal Denominator having as many o's as r has Figures, is a greater Number than r ; and so a Fraction whose Numerator is 1, and its Denominator that decimal one, is less than $\frac{a}{r}$, because the Denominator is greater, and the Numerator not. Lastly, since, as has been shewn, 1 added to the last Figure of the Root would make it exceed a true Root; therefore, in whatever Place of Decimals the last Figure of the Root stands, the whole does not want of a true Root to the given Number, an Unit of the Value of that Place, and consequently, if the Denominator of that last Place is a Number greater than r , the Root is within $\frac{1}{r}$ of a true Root, because it's within a lesser Fraction, and much more is it within $\frac{a}{r}$, which is greater than either of the former. Example: Let r be a Number of three Figures; if the Root have three decimal Places whereby the Denominator is 1000, the Root is within $\frac{1}{1000}$, which is less than any proper Fraction whose Denominator is a Number of three Figures; so in the preceding Example 19.672 is within $\frac{1}{1000}$ of a perfect Root to 387.

II. For the Roots of Fractions.

The Approximation of the Roots of Fractions is performed thus: Let that Root which the particular Rules for Square and Cube, or the general Rule following these, prescribes to be extracted, be carried on to what Length of decimal Places you please, and then divide it by the Denominator of the given Fraction, if you chuse the first Method of these Rules; and thus you have a Root still less, but approaching to a perfect one: But if you chuse the 2d Method of the Rule, divide the given Numerator by that Root, and you have a Root approaching, but still excessive; and the further the Approximation of that Root is carried, the truer will each of these fractional Roots be.

Example. For the Square Root of $\frac{13}{24}$, I take $13 \times 24 = 312$, whose Root to 2 Places of decimals is 17.66, which divided by 24 Quotes $\frac{1766}{2400} = \frac{883}{1200}$, less than a true Root; or it is $13 \div 17.66 = \frac{1300}{1766} = \frac{650}{883}$, greater than a true Root.

DEMON. The Reason is manifest from the preceding Rules; for the Square Root of $\frac{A}{B}$ is $\overline{AB^{\frac{1}{2}}} \div B$, or $A \div \overline{AB^{\frac{1}{2}}}$. Universally, the n Root of $\frac{A}{B}$ is $\overline{AB^{\frac{1}{n}}} \div B$, or $A \div \overline{BA^{\frac{n-1}{n}}}$ if these Roots are perfect; and if they are not, yet by approximating them we make the fractional Root also truer, tho' never perfect.

COROLL. Tho' a given Integer is not a perfect Power of any Order, yet there is a greatest Power of that Order, which is a lesser Number than the given one; and also there is a least Power of the same Order, which is a greater Number than the given: But in Fractions there is no such greatest and least Power; because we can find new Roots increasing for ever, or decreasing, yet so as the Powers are still less or greater than the given Number.

SCHOLIUM. There remains one curious Problem relating to the Extraction of Roots, which goes a little deeper into the Algebraick Art than at first I designed: but without it, I found I must omit several other curious things: and since among several ways of solving this Problem, there is one that arises very easily and naturally from the Consideration of Square Numbers, especially the Square of a Binomial Root (already sufficiently explained) therefore I was determined to give it a place here.

P R O B L E M VI.

Having the Sum or Difference of any Square Number, and a certain Multiple of the Root; also having the Multiplier of the Root; to find the Root. Thus: Suppose $R = a^2 + ac$, or $R = a^2 - ac$, or $R = ac - a^2$. Then if the Numbers expressed by R and c are given, we can find the Number expressed by a by the following Rules.

C A S E I.

When the Sum and Multiplier are given to find the Root, *i. e.* if $R = a^2 + ac$; and R, c are given to find a .

Rule. To the Sum add the Square of half the Multiplier, (or a 4th of the Square of the Multiplier.) Extract the Square Root of this Sum; and from it subtract half of the Multiplier, the Remainder is the Root sought. Which Rule is expressed in Characters thus:

$$a = R + \frac{c^2}{4} - \frac{c}{2}.$$

Exam-

Example. $R=21$, $c=4$; then is $a=3$; for $4 \times 4=16$, whose 4th is 4; then $21 + 4=25$, whose Square Root is 5, from which take 2 (= the half of the Multiplier 4) the Remainder is 3 the Root. *Proof:* $3 \times 3=9$, $3 \times 4=12$, and $12 + 9=21$.

DEMON. Since by Supposition, $R=a^2+ac$; add $\frac{cc}{4}$ to each Side, then $R + \frac{cc}{4} = a^2 + ac + \frac{cc}{4}$; which last Expression is a compleat Square, whose Root is $a + \frac{c}{2}$: therefore $a + \frac{c}{2} = \sqrt{R + \frac{cc}{4}}$ (Ax. 1.) and subtracting $\frac{c}{2}$ from both, it is, $a = \sqrt{R + \frac{cc}{4}} - \frac{c}{2}$. Which is the Rule.

CASE II.

If the Difference and Multiplier are given, to find the Root;
Here there are two Rules, according as the Square or Multiple is supposed to be greatest.

1. Suppose the Square greater than the Multiple, i.e. $R=a^2-ac$.

RULE. To the Difference, add the 4th of the Square of the Multiplier; and to the Square Root of the Sum, add half the Multiplier: this Sum is the Root sought. Thus:

$$a = \sqrt{R + \frac{cc}{4}} + \frac{c}{2}.$$

Example. $R=28$, $c=3$; then is $a=7$: for $\frac{cc}{4} = \frac{9}{4} = 2\frac{1}{4}$, and $R + \frac{cc}{4} = 28 + 2\frac{1}{4} = 30\frac{1}{4} = \frac{121}{4}$, whose Square Root is $\frac{11}{2}$ or $5\frac{1}{2}$; to which add $\frac{3}{2}$ or $1\frac{1}{2}$, the Sum is 7.

Proof. $7 \times 7=49$, and $3 \times 7=21$; then $49 - 21=28$.

DEMON. Since $R=a^2-ac$, add $\frac{cc}{4}$ to both Sides; then is $R + \frac{cc}{4} = a^2 - ac + \frac{cc}{4}$.

Which last Expression is the Square of $a - \frac{c}{2}$. Wherefore $a - \frac{c}{2} = \sqrt{R + \frac{cc}{4}}$; and adding $\frac{c}{2}$ to both Sides, it is $a = \sqrt{R + \frac{cc}{4}} + \frac{c}{2}$. Observe, Tho' $a^2 - ac + \frac{cc}{4}$ is the Square

either of $a - \frac{c}{2}$, or $\frac{c}{2} - a$, yet we cannot here use $\frac{c}{2} - a$; for if a is less than $\frac{c}{2}$, a^2 is less than ac , contrary to Supposition.

2. Suppose the Multiplier greater than the Square, i.e. $R=ac-a^2$.

RULE. From the 4th of the Square of the Multiplier subtract the given Difference; (which cannot exceed the Multiplier, if the Problem is possible); then extract the Square Root of the Remainder; and either add it to, or subtract it from half the Multiplier, (which is greater than the other, if the Problem is possible); the Sum or Difference will either of

them solve the Problem. Thus: $a = \frac{c}{2} + \sqrt{\frac{cc}{4} - R}$, or also $a = \frac{c}{2} - \sqrt{\frac{cc}{4} - R}$.

Example. $R=6$, $c=5$; then is $a=3 = \frac{5}{2} + \sqrt{\frac{25}{4} - 6}$, or $\frac{5}{2} + \frac{1}{2} = \frac{6}{2}$.

Proof. $ac=15$, and $ac-a^2=15-9=6=R$. Also, $a=2 = \frac{5}{2} - \frac{1}{2} = \frac{4}{2}$. *Proof.*

$ac-a^2=10-4=6=R$.

C c

DEMON.

DEMON. Since $R = ac - a^2$. Subtract each of these from $\frac{cc}{4}$, then is $\frac{cc}{4} - R = \frac{cc}{4} - ac + a^2$; which left is the Square either of $a - \frac{c}{2}$, or $\frac{c}{2} - a$. Wherefore $a - \frac{c}{2}$, or $\frac{c}{2} - a$ (according as a is greater or lesser than $\frac{c}{2}$) is $= \sqrt{\frac{cc}{4} - R}$. Hence, in the 1st Case, or taking $a - \frac{c}{2}$; by adding $\frac{c}{2}$ to both Sides, it is $a = \frac{c}{2} + \sqrt{\frac{cc}{4} - R}$; and taking $\frac{c}{2} - a$, add $a - \frac{c}{2}$ to both Sides, it is $a = \frac{c}{2} - \sqrt{\frac{cc}{4} - R}$.

There remains yet to be demonstrated, That $\frac{cc}{4}$ can never be less than R , if the Problem is possible; and that $\frac{c}{2}$ is greater than $\sqrt{\frac{cc}{4} - R}$. Now it is plain, that the Solution is impossible, according to this Rule, if R is greater than $\frac{cc}{4}$; and that if R does not exceed $\frac{cc}{4}$, one of the Solutions is good. But to shew that the Problem will always necessarily have the two Solutions explained, it must be shewn that R cannot exceed $\frac{cc}{4}$, when it is $= ac - a^2$, and then the other Part will easily follow.

To demonstrate this, we must first observe, That a may be either greater or lesser than $\frac{c}{2}$ consistently enough with $R = ac - aa$. For this requires no more than that ac be greater than aa , which requires again that a be less than c ; consequently, whether a be greater or less than $\frac{c}{2}$, providing it be less than c , (as it may be) ac will be greater than aa . Again; Whether we take $a - \frac{c}{2}$ or $\frac{c}{2} - a$, the Square of it is $a^2 - ac + \frac{cc}{4} = \frac{cc}{4} - ac + a^2$, which is also $= \frac{cc}{4} - ac + aa = \frac{cc}{4} - R$, (because $ac - aa = R$.) But the Root being real or positive, so must the Square be; i. e. $\frac{cc}{4}$ is greater than R . Or, if $a = \frac{c}{2}$, then $a^2 = \frac{cc}{4}$, and $2aa = c$; also $2aa = ac$; consequently $ac - aa = 2aa - aa = aa$; and $\frac{cc}{4} = ac - aa = R$. So that R can never be greater than $\frac{cc}{4}$, tho' it may be either equal or less. And observe, if they are equal, then there is but one Solution, viz. $a = \frac{c}{2}$; for here both the Solutions coincide.

For the second thing, viz. that $\frac{c}{2}$ is greater than $\sqrt{\frac{cc}{4} - R}$; consider that $\frac{cc}{4}$ is greater than $\frac{cc}{4} - R$, and consequently $\frac{c}{2}$ (the Square Root of $\frac{cc}{4}$) is greater than $\sqrt{\frac{cc}{4} - R}$. Or, we have this in the very Supposition; for, by the first Part of the Demonstration of this Rule, the Value

Value of a is $= \frac{c}{2} - \sqrt{\frac{cc}{4} - R}$, upon that very Supposition that $\frac{c}{2}$ is greater than a ;
whence it was shewn that $\sqrt{\frac{cc}{4} - R} = \frac{c}{2} - a$, and consequently $\frac{c}{2}$ greater than $\sqrt{\frac{cc}{4} - R}$.

SCHOLIUMS.

1. If the Difference of the Square and Multiple of the Root is given, without determining which of them is greatest, then we must try both Rules.

2. This Problem is what the *Algebraists* call, *Extracting the Root of an affected Square*, (i. e. wherein the Number given is the Sum or Difference of a Square, and a certain Multiple of the Root; whose Multiplier is also given) The Solutions explained are all that are real and positive; yet the Algebraick Art considers two Roots or Solutions in every Case: But the Roots that I have not explained are only negative and imaginary; and to say any thing farther about them, were to exceed the Limits prescribed to this Work; and for the same Reason I am obliged to speak nothing of extracting the Roots of higher Powers that are affected.

CHAP. III.

The Arithmetick of SURDS.

WHAT a *Surd* is has been already explained: It has been demonstrated that every Number has not a perfect and determinate Root; but yet that we can find an Approximate Root within any assignable Difference of a true and compleat one; so that it may be truly said, that the Quantity which hinders any Number from being a compleat Power of any kind, is infinitely little; or that a Quantity infinitely little (or less than any assigned one) being taken from the Quantity expressed by any given Number, the Remainder is a Quantity expressible by a Number (of the same Parts) which is a true Power of the Order proposed; with this Difference, that it will be a fractional Power and not an integral. Now since *Surds*, or indeterminate Roots, can be determined infinitely near; and since the indeterminate Series goes on by a certain Law or Condition, it may be conceived as some whole and compleat thing of its own kind; and therefore, taking *Surds* under the general Expression of Roots, as $N^{\frac{1}{n}}$, we may apply all the *Theory* of Chap. I. and all the *Operations* of Arithmetick to them, as if they were determinate: For thus we can form general Ideas of Sums, Differences, Products and Quotes of *Surds*, imagined under the Notion of compleat Quantities of their own kind, the same way as we do of rational or determinate Roots expressed after the same general manner. And hereby we can discover certain Connections and Relations of Quantities thus represented, which may lead us to some other particular Truths we would discover.

It's true, indeed, that as to any actual Operation with such Roots it can only be made in an imperfect manner, by way of Approximation; yet since we can approximate or determine the Root so far, that taking it for true and compleat, the Error it can make in any Operation or Conclusion shall be within any assignable Difference of what it would be if the compleat Value of the *Surd* could possibly be determined and used in the Operation: Therefore our arguing with them as we do with rational and determinate Roots, is so far at least just and conclusive; and is indeed absolutely so, taking them in general and abstractly.

To illustrate this by a few *Examples*: The Sum of $\sqrt{8}$ and $\sqrt{12}$ may be expressed in general $\sqrt{8} + \sqrt{12}$, whatever these are in themselves; and if we would apply this by an actual Operation, then we can approximate each of these Roots so near, that their Sum shall want less than any assigned Difference of what it would be if the Roots could be determined.

Example 2. To multiply $\sqrt{8}$ by 3, it may be expressed thus, $3 \times \sqrt{8}$; and by Approximation we can find a determinate Number for $\sqrt{8}$, which multiplied by 3, the Product shall want less than any assigned Difference of what it would be if the Root could be completely determined.

Example 3. The Product of $\sqrt{8}$ and $\sqrt{6}$ may be expressed $\sqrt{8} \times \sqrt{6}$; and by Approximation we can take $\sqrt{8}$ and $\sqrt{6}$ so near, that multiplying them together at every Step, the Products shall still increase and come within any assigned Difference of what it would be were the Roots determined.

Again: Tho' Surds can never be reduced to determinate Numbers (for then they were not Surds), yet in many Cases their Sums, Differences, Products and Quotes can be expressed after different ways (by means of the *Theory* explained in *Chap. I.*), which are more or less simple and convenient; so that what by the more general Rules can be expressed only by Signs of Addition, &c. may be expressed more simply, either by one Surd, or by an Expression partly surd, partly rational, and in some Cases altogether rational. Now to this tends the more particular Practice or Arithmetick of Surds; which depending upon certain different Forms in which the same Surd may be expressed, therefore the first thing to be explained is *The Reduction of Surds*; the Demonstration of which depends upon the *Theorems* in *Chap. I.* applied to *Surds*.

Observe also, That all the following Practice is equally applicable to rational Roots expressed in the general radical Form; for when we take general Expressions they comprehend all possible Cases; and the Practice proposed is often as convenient with respect to Rationals as Surds, because it's convenient sometimes to express even rational Numbers in this radical Form; and therefore, tho' it's commonly called the *Arithmetick of Surds*, it were as proper to call it the *Arithmetick of Radicals*.

Reduction of Surds (or Radicals).

CASE I. To express any Number in Form of a Surd (*i. e.* in a radical Form).
RULE: Raise the given Number to the Power of the Surd, and then apply the Surd Index, thus; $8 = 64^{\frac{1}{2}}$, for $8 \times 8 = 64$. Universally, $A = \overline{A^n}^{\frac{1}{n}}$.

The Reason is manifest from the *Definitions*, and *Ax. I.*

CASE II. To reduce a Surd with a Mixt Index (*i. e.* whose Numerator is greater than 1) to another, having a simple radical Index (*i. e.* whose Numerator is 1.) **RULE:** Involve the Number given to a Power whose Index is the Numerator of the Mixt Index, and to the Number found apply the Denominator radically. *Example:* $8^{\frac{2}{3}} = 64^{\frac{1}{3}}$; for $8^2 = 64$. Universally, $A^{\frac{n}{r}} = \overline{A^n}^{\frac{1}{r}}$.

The *Demonstration* of this is plainly in the Definition: For $A^{\frac{n}{r}}$ expresses the r Root of the n Power (which is also the n Power of the r Roots, by *Theor. X.*)

CASE III. To reduce two Unlike Surds to Like: *i. e.* having two unlike Surds of the same or different Numbers, to find other two Surds equal respectively to the given ones, but having the same Index, and that also the least possible; and such too, that the Numbers under the common Index be the least possible;
RULE.

RULE. Reduce their Indexes to one common Denominator by the Rules of Fractions. Again, find the greatest common Measure to both the new Numerators, (*i.e.* the greatest Number which will divide them both without a Remainder,) by the Method taught in the Reduction of the Numerator and Denominator of a Fraction to their least Terms; make that common Measure the common Numerator to the common Denominator before found: The lowest Terms of this Fraction is the common Index sought.

Again; Divide the new Numerators mentioned by their greatest common Measure, and mark the Quotes; then involve each given Number to a Power whose Index is the respective Quote; and that is the Number to which if the common Index is applied the Case is compleatly solved.

Example: To reduce $8^{\frac{1}{2}}$ and $15^{\frac{1}{3}}$ to Like Surds, with the other Conditions proposed.

1. They are $8^{\frac{1}{2}}$ and $15^{\frac{1}{3}}$, by reducing the Indexes $\frac{1}{2}$ and $\frac{1}{3}$ to one Denominator; Then the greatest common Measure of the Numerators 2, 3, is 1, and the common Index is $\frac{1}{6}$; and to have Numbers to which it must be applied, I raise 8 to the 3d Power, and 15 to the 2d, (for here the common Measure of 3 and 2 is 1, which makes the Quotes the same,) these Powers are 512, 225; wherefore the Surds sought are $512^{\frac{1}{6}} = 8^{\frac{1}{2}}$, and $225^{\frac{1}{6}} = 15^{\frac{1}{3}}$.

Example 2. To reduce $4^{\frac{2}{3}}$ and $5^{\frac{4}{7}}$: They are first $4^{\frac{14}{21}}$, $5^{\frac{12}{21}}$, then the greatest common Measure of 14, 12, is 2; and so $\frac{2}{21}$ is the common Index, which is in its least Terms. Again; The Numerators 14, 12, divided by their greatest Measure 2, the Quotes are 7, 6, and $4^7 = 16384$, $5^6 = 15625$. Then lastly, $16384^{\frac{2}{21}} = 4^{\frac{2}{3}}$, $15625^{\frac{2}{21}} = 5^{\frac{4}{7}}$.

Example 3. To reduce $3^{\frac{2}{3}}$ and $4^{\frac{1}{3}}$: They are first $3^{\frac{4}{6}}$, $4^{\frac{2}{6}}$, and the greatest Measure of the Numerators 3, 6, being 3, the common Index is $\frac{3}{18} = \frac{1}{6}$ in its least Terms; then 3, 6, divided by 3, the Quotes are 1, 2; and $3^1 = 3$, $4^2 = 16$: Wherefore, lastly, $3^{\frac{1}{6}}$, $16^{\frac{1}{6}}$ are the Surds sought.

DEMON. Let $A^{\frac{r}{s}}$, $B^{\frac{n}{t}}$ be any two Surds, (where, if r or n are 1, the Surds are simple.) These are first equal to $A^{\frac{ru}{su}}$, $B^{\frac{nt}{st}}$ by Reduction of the Indexes to one Denominator, (*Theor.* XI. *Ch.* I.) Suppose m to be the common Measure to ru , nt , and let the Quotes be $ru \div m = x$; $nt \div m = y$; so that $ru = mx$, and $nt = my$: Then the Surds are $A^{\frac{mx}{su}}$, $B^{\frac{my}{st}}$, that is, (by *Theor.* XII.) $\overline{A^x}^{\frac{m}{su}}$, $\overline{B^y}^{\frac{m}{st}}$; which is exactly the Expression of the Rule, supposing m the greatest common Measure of ru , nt , and $\frac{m}{su}$ to be in its least Terms; or if it's not in lowest Terms, yet its lowest Terms being put in its Place makes an equivalent Expression (*Theor.* XI.) Observe also, that tho' m is not the greatest common Measure of the new Numerators ru , nt , yet we have the Surds reduc'd to Like Surds, tho' not in the lowest Expressions; which it's plain will then, and then only happen when m is the greatest common Measure.

SCHOLIUMS.

1. It will be the same thing if we first reduce the given Surds to simple Indexes, if they are Mixt, and then reduce these new Indexes to one Denominator, and go on with the rest as in the Rule.

I

2. When

2. When there are more Surds proposed, the Operation and the Reason of it is the same; except that we have not yet learn'd how to find the greatest common Measure to 3 or more Numbers (which you'll find in *Book IV. Ch. I.*) and therefore, till that be learn'd, we must be content to reduce the Surds to Likes, tho' not in their lowest Terms, by using 1 as a common Measure, which makes the Dividend and Quote the same.

CASE IV. To reduce a Surd having a simple Index to lower Terms; *i. e.* to an equivalent Expression in which there is a similar Surd of a lesser Number multiplied into some rational Number.

RULE. Among the Numbers greater than 1, which measure the given Number, (or Surd Power) seek one which is a similar and rational Power, by which divide the given Number: Take the Quote, and to it apply the given Index; and multiply that Root by the Root of the Divisor: This Product is the Expression sought.

Example 1. $8^{\frac{1}{2}} = 2 \times 2^{\frac{1}{2}}$; for $8 \div 4 = 2$, and $4^{\frac{1}{2}} = 2$.

Example 2. $648^{\frac{1}{3}} = 2 \times 81^{\frac{1}{3}} = 3 \times 24^{\frac{1}{3}}$; for $648 \div 8 = 81$, and $8^{\frac{1}{3}} = 2$; whence, by the Rule, the 1st Solution is $2 \times 81^{\frac{1}{3}}$. Again; $648 \div 27 = 24$, and $27^{\frac{1}{3}} = 3$; whence the 2d Solution is $3 \times 24^{\frac{1}{3}}$.

DEMON. Suppose $A \div D^n = B$, so that $A = B \times D^n$, then is $A^{\frac{1}{n}} = \overline{B \times D^n}^{\frac{1}{n}}$, (*Ax. I.*) and $\overline{B \times D^n}^{\frac{1}{n}} = D \times B^{\frac{1}{n}}$, (*Theor. III.*) which is precisely conform to the Rule; A representing the given Number.

SCHOLIUMS.

1. If the Power by which we measure the given Number is the greatest Like Power which measures it, then we find the lowest Terms of the given Surd.

2. As to the finding the Numbers that measure any given Number, you'll have it more particularly explained in *Book IV. Chap. I.* Here we suppose these to be given; because from the Nature of the Thing this Rule for finding them is obvious, *viz.* To try all the Numbers not exceeding the half of the given Number; for all of these which measure it, together also with the Quotes, make all the Numbers that measure it. But unless these Measures that serve the present Problem are obvious, the finding them out is more Trouble than is always necessary.

3. If the given Surd has a mixt Index, the same kind of Reduction may be performed by reducing it first to a Surd with a simple Index; and then applying the present Rule. And again; If the Number under the radical Sign in the Answer thus found, is a rational Power of the Order expressed by the Numerator of the given Index, then by taking the Root of it we may also reduce the whole to a Surd with the given mixt Index. Thus, $192^{\frac{2}{3}}$ is first $= 36864^{\frac{1}{3}}$ ($= 192^{\frac{2}{3}}$); which again, reduced is $= 16 \times 9^{\frac{1}{3}}$; for $16^3 = 4096$, and $36864 \div 4096 = 9$: And because 9 is a Square, therefore $9^{\frac{1}{3}} = 3^{\frac{2}{3}}$; and hence $16 \times 9^{\frac{1}{3}} = 16 \times 3^{\frac{2}{3}} = 192^{\frac{2}{3}}$.

COROLL. Hence we see plainly, That one similar Surd may be a Multiple or aliquot Part of another. But observe that in applying this to Practice, all we can make of it is, That the greater Surd approximate to a certain degree, and divided by the other approximate to the same degree, the Quote will be within a certain Difference of that Number, which, by this Reduction, appears to be the Quote: But being approximate nearer and nearer *in infinitum*, the Quote will be nearer *in infinitum* to that other, which we here call the True and Complete Quote. But if the same Dividend be divided by any

any other Surd or Number whatever, the Quote can be brought to exceed that true Quote, or will never be brought within an assignable Difference of it; and therefore it's justly called the true Quote of these two Surds.

CASE V. To reduce any two Surds to Expressions, having a common Surd; *i. e.* to Expressions that are Products of rational Numbers into a common Surd.

RULE. Reduce the given Surds to the same simple Index, if they are not so already (by Case 3.): Then find the greatest common Measure of the Powers (or Numbers under the radical Signs); and taking the Quotes, examine by Extraction if they are rational and similar Powers of the Order expressed by the Denominator of the common simple Index; if they are, their Roots are the rational Numbers sought; and the surd Root of the common Measure is the surd Part sought: But if these Quotes are not such similar Powers, the Question is impossible.

Example: To reduce $12^{\frac{1}{2}}$ and $27^{\frac{1}{2}}$: The greatest common Measure of 12, 27, is 3, and the Quotes are 4, 9. which being rational Squares, I take their Roots 2, 3, and multiply them into the common Surd $3^{\frac{1}{2}}$, and the Expressions sought are $2 \times 3^{\frac{1}{2}} = 12^{\frac{1}{2}}$, and $3 \times 3^{\frac{1}{2}} = 27^{\frac{1}{2}}$.

DEMONSTR. Let $A^{\frac{1}{n}}$, $B^{\frac{1}{n}}$ be the given Surds (or the Expressions to which they are reduced): Suppose $A \div m = a^n$, and $B \div m = b^n$, so that $A = m \times a^n$, and $B = m \times b^n$; then $A^{\frac{1}{n}} = \sqrt[n]{m \times a^n}$ (Ax. I.) $= a \times m^{\frac{1}{n}}$, (Theor. I. Cor.) Also $B^{\frac{1}{n}} = \sqrt[n]{m \times b^n}$, $= b \times m^{\frac{1}{n}}$, which is exactly according to the Rule, supposing m to be any common Measure: And the Reason why it's in the Rule called the greatest common Measure, is, because if the greatest will not quote similar rational Powers, none of the other common Measures will; which remains to be demonstrated. Thus; take the given Surd Powers fractionwise, $\frac{A}{B}$; This is not an immediate fractional Power of the Order n , because by Supposition, neither A or B are rational Powers of that Order; but if any other Fraction equivalent to $\frac{A}{B}$ is an immediate Power of that Order, the least Terms of $\frac{A}{B}$ will be so; and if the least Terms are not so, no other Terms can be so, (as has been demonstrated in the Rule for Extracting the Roots of Fractions) *i. e.* if A, B , being divided by their greatest common Measure, do not give for Quotes similar rational Powers of the Order n , neither can their Quotes by any other common Measure do so.

SCHOLIUMS.

1. The greatest common Measures quoting similar rational Powers, is a certain Character of the Problems being possible, tho' none of the other common Measures should make such Quotes; but if any of these others do so, these would make so many different Solutions to the Problem; in which this Difference is to be observed, that the lesser the common Measure is which we use, the lesser Terms will the Solution be in, as to the Surd Part: And the Reason why we chuse the greatest Measure in the Rule is, because that tho' from any other Measure's giving Quotes which are rational Powers we are sure that the Problem is possible, yet we can conclude it impossible from no other but the greatest common Measure giving Quotes which are not Like Powers.

2. If the two Surd Powers are Fractions, then reduce them to any common Denominator, and if the new Numerators are reducible according to this Rule, so are the given Surds.

Surds. One Example will shew this: Suppose $\sqrt[12]{\frac{12}{15}}$, $\sqrt[3]{\frac{3}{15}}$, the Numerators 12, 3, taken radically, viz. $12^{\frac{1}{2}}$, $3^{\frac{1}{2}}$, are reducible to these, $2 \times 3^{\frac{1}{2}}$, $1 \times 3^{\frac{1}{2}}$, or $3^{\frac{1}{2}}$; wherefore the given Surds are reduced to these, $2 \times \sqrt[15]{\frac{3}{15}}$, and $\sqrt[15]{\frac{3}{15}}$.

Again; If either the Numerators or Denominators of two Fractions, affected with a simple radical Sign, or reduced to that State, are rational Powers of the Order expressed by the Denominator of the Index. The Fractions need not be reduced to a common Denominator; for we need only examine if the other Terms are reducible to a common Surd Power: Thus; Suppose $\sqrt[50]{\frac{50}{16}}$, $\sqrt[72]{\frac{72}{25}}$: Here the Denominators, 16, 25, are Squares, whose Roots are 4, 5. Again; $50^{\frac{1}{2}} = 2 \times 25^{\frac{1}{2}} = 5 \times 2^{\frac{1}{2}}$. Also $72^{\frac{1}{2}} = 2 \times 36^{\frac{1}{2}} = 6 \times 2^{\frac{1}{2}}$, whence it's plain that the given Surds are $2^{\frac{1}{2}} \times \frac{5}{4}$, $2^{\frac{1}{2}} \times \frac{6}{5}$. And had the given Surds been $\sqrt[16]{\frac{16}{59}}$, $\sqrt[25]{\frac{25}{72}}$, the Solution is $\frac{4}{5} \times \sqrt[1]{\frac{1}{2}}$, $\frac{5}{6} \times \sqrt[1]{\frac{1}{2}}$; for $16^{\frac{1}{2}} = 4$, and $50^{\frac{1}{2}} = 5 \times 2^{\frac{1}{2}}$, then is $\sqrt[50]{\frac{50}{16}} = \frac{4}{5 \times 2^{\frac{1}{2}}} = \frac{4}{5} \times \frac{1}{2^{\frac{1}{2}}}$, or $\frac{4}{5} \times \sqrt[1]{\frac{1}{2}}$, and so of the other.

3. This Case is commonly called *Finding*, if two Surds are commensurable; i. e. if they have a common Measure, or if there is any Surd which measures or is an aliquot Part of each of them; whereby they are reducible to Expressions which are the Products of that common Surd into the respective Quotes. Observe also, That the Measure of a Surd must be a Surd, which is manifest; for if any rational Number should measure a Surd, or be an aliquot Part of it, then that aliquot Part and its Denominator (or the Measure and Quote) would produce the Dividend, i. e. two rational Numbers would produce a Surd, which is impossible.

The Use of these Reductions in the common Operations of Addition, &c. I shall briefly shew thus:

In Addition and Subtraction of Surds.

If one Surd is to be added to or subtracted from another, and if they are commensurable, i. e. reducible to a common Surd, by Case 5. this Reduction being made, or if the given Expressions are of this kind, the Sum or Difference of the rational Parts multiplied into the common Surd Part is the Sum or Difference sought, in a more simple and convenient Form than connecting the given Numbers by the general Signs of Addition and Subtraction, which is the general Rule for all other Cases.

Example 1. $8^{\frac{1}{2}} + 50^{\frac{1}{2}} = 2 \times 2^{\frac{1}{2}} + 5 \times 2^{\frac{1}{2}} = 7 \times 2^{\frac{1}{2}}$.

Example 2. $54^{\frac{1}{2}} - 16^{\frac{1}{2}} = 3 \times 2^{\frac{1}{2}} - 2 \times 2^{\frac{1}{2}} = 3 - 2 \times 2^{\frac{1}{2}} = 2^{\frac{1}{2}}$.

The Sum or Difference of two Square Roots may be also express'd. Thus: Take the Surd Powers, or Numbers under the radical Sign; to the Square Root of double their Product, add their Sum, or subtract that Root from this Sum; the Square Root of this Sum or Difference expresses the Sum or Difference sought. *Example:* $5^{\frac{1}{2}} + 3^{\frac{1}{2}} = 8 + 30^{\frac{1}{2}}$, and $5^{\frac{1}{2}} - 3^{\frac{1}{2}} = 8 - 30^{\frac{1}{2}}$.

DEMON.

DEMON. Suppose $A^{\frac{1}{2}} = a$, $B^{\frac{1}{2}} = b$; then is $ab = A^{\frac{1}{2}} \times B^{\frac{1}{2}} = \overline{AB}^{\frac{1}{2}}$ (*Theor. 3.*) and $A^{\frac{1}{2}} + B^{\frac{1}{2}} = a + b = \overline{a^2 + b^2 + 2ab}^{\frac{1}{2}}$. Also $A^{\frac{1}{2}} - B^{\frac{1}{2}} = a - b = \overline{a^2 + b^2 - 2ab}^{\frac{1}{2}}$, which is exactly according to the Rule.

For Multiplication and Division of SURDS.

If one Surd is to be multiplied or divided by another; then if they are unlike, reduce them to Likes, and examine if they are commensurable, *i. e.* reducible to Expressions, wherein the same common Surd is multiplied into rational Numbers; and if it is so, multiply or divide the rational Parts, the Product multiplied again into the Square of the common Surd is the Product sought: so that if the given Surds are Square Roots, the Product is rational. But in Division the Quote of the rational Parts alone is the Quote sought; which is therefore rational.

Example. $72^{\frac{1}{2}} \times 32^{\frac{1}{2}} = 48$. For $72 = 8 \times 9$, and $9^{\frac{1}{2}} = 3$; therefore $72^{\frac{1}{2}} = \overline{8 \times 9}^{\frac{1}{2}} = 9^{\frac{1}{2}} \times 8^{\frac{1}{2}} = 3 \times 8^{\frac{1}{2}}$. Again, $32^{\frac{1}{2}} = \overline{4 \times 8}^{\frac{1}{2}} = 4^{\frac{1}{2}} \times 8^{\frac{1}{2}} = 2 \times 8^{\frac{1}{2}}$: so that $72^{\frac{1}{2}} \times 32^{\frac{1}{2}} = 3 \times 8^{\frac{1}{2}} \times 2 \times 8^{\frac{1}{2}} = 3 \times 2 \times 8^{\frac{1}{2}} \times 8^{\frac{1}{2}} = 6 \times 8 = 48$. And $72^{\frac{1}{2}} \div 32^{\frac{1}{2}} = 3 \times 8^{\frac{1}{2}} \div 2 \times 8^{\frac{1}{2}} = 3 \div 2$.

DEMON. Suppose $A^{\frac{1}{n}} = a \times R^{\frac{1}{n}}$, and $B^{\frac{1}{n}} = b \times R^{\frac{1}{n}}$; then $A^{\frac{1}{n}} \times B^{\frac{1}{n}} = ab \times R^{\frac{1}{n}} \times R^{\frac{1}{n}} = ab \times R^{\frac{2}{n}}$. Wherefore if $n = 2$, the Product is abR . Also $A^{\frac{1}{n}} \div B^{\frac{1}{n}} = a \times R^{\frac{1}{n}} \div b \times R^{\frac{1}{n}} = a \div b$.

S C H O L I U M S.

1. To multiply similar Surds: If we multiply the Surd Powers, and apply the same Index to that Product; this expresses the Product sought more simply, than by the general Sign of Multiplication. Thus: $A^{\frac{1}{n}} \times B^{\frac{1}{n}} = \overline{AB}^{\frac{1}{n}}$ (*Theor. 3.*) Again; if this is reducible, bring it to lowest Terms, and you'll have in many Cases the same Product that the preceding Rule brings out; and it's always the best we can make of it, when the given Surds are not commensurable. In the preceding Example, $72^{\frac{1}{2}} \times 32^{\frac{1}{2}} = \overline{72 \times 32}^{\frac{1}{2}} = \overline{2304}^{\frac{1}{2}} = 48$.

2. If a rational Number is multiplied into a Surd, it may be sometimes convenient to express it altogether radically; for which you have a Rule in *Theo. 3. Cor.* Thus: Raise the rational Part to the Power whose Index is the Denominator of the surd Part, and multiply this Power into the surd Power; then apply the radical Index. Exam. $A \times B^{\frac{1}{n}} = \overline{A^n B}^{\frac{1}{n}}$, and $A \times B^{\frac{1}{n}} = \overline{A^n B^{\frac{1}{n}}}$, for $B^{\frac{1}{n}} = \overline{B^{\frac{1}{n}}}$.

3. If a Surd and rational Number are multiplied; and if the Surd is reducible to lower Terms, the whole Product is so. Thus: $6 \times 45^{\frac{1}{2}} = 6 \times 3 \times 5^{\frac{1}{2}} = 18 \times 5^{\frac{1}{2}}$; for $45 = 9 \times 5$, and $45^{\frac{1}{2}} = 9^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 3 \times 5^{\frac{1}{2}}$.

You may apply the same Observations to Division. So for the 1st, $72^{\frac{1}{2}} \div 32^{\frac{1}{2}} = \overline{72 \div 32}^{\frac{1}{2}} = \overline{9 \div 4}^{\frac{1}{2}}$, or $\frac{9}{4}^{\frac{1}{2}} = \frac{3}{2}$.

And from this Example, wherein the Product or Quote becomes rational, we have a farther remarkable Proof of the Reasonableness and Usefulness of our treating Surds, and working with them in all respects as with Rationals or compleat Roots; for if any other Number than 48 is supposed to be the Product of $72^{\frac{1}{2}} \times 32^{\frac{1}{2}}$, we can prove it to be false.

D d

Thus:

Thus: Since $72^{\frac{1}{2}}$ and $32^{\frac{1}{2}}$ can be Approximate *in infinitum*, the Products of these approximate Roots will grow *in infinitum* towards a certain Limit; which must necessarily be $\overline{72 \times 32^{\frac{1}{2}}} = 48$; because if $72^{\frac{1}{2}}$ and $32^{\frac{1}{2}}$ are both rational, then is $\overline{72 \times 32^{\frac{1}{2}}}$ rational, being equal to $72^{\frac{1}{2}} \times 32^{\frac{1}{2}}$ (*Theor. 3.*) And tho' $72^{\frac{1}{2}}$, $32^{\frac{1}{2}}$ are surd, yet being infinitely approximable, their Product will grow infinitely near to $\overline{72 \times 32^{\frac{1}{2}}} = 48$; which is therefore the true Limit or compleat Value of $72^{\frac{1}{2}} \times 32^{\frac{1}{2}}$, beyond which it cannot possibly grow; nor can it be supposed less, because we can approximate $72^{\frac{1}{2}}$ and $32^{\frac{1}{2}}$ so far, that the Product shall exceed any assignable Number less than $48 = \overline{72 \times 32^{\frac{1}{2}}}$; for else they were not infinitely approximable, as is supposed and demonstrated.

C H A P. IV.

Containing several THEOREMS relating to the Powers of Numbers.

IN the following *Theorems* and *Corollaries* are comprehended all the Propositions of the *Second Book* of EUCLID that are applicable to Numbers, excepting four, which are in effect already explained in this Work; but that you may see them all in this Place, I briefly repeat these four.

1. If one Number A, (or Line, as it is in *Euclid, Book II. Theor. 1.*) is multiplied severally into all the Parts of another $B = a + b + c$, &c. the Sum of the Products is the Product of the two Wholes; thus $Aa + Ab + Ac$, &c. $= AB$. This you have already in *Lemma 3. Ch 5. Book I.* which, observe, is equally applicable to Fractions and Integers.

2. If any Number is multiplied into all its own Parts severally, the Sum of the Products is equal to the Square of the Whole; which is the Consequence of the last; thus, if $N = a + b$, then is $N^2 = Na + Nb$ (*Euclid, Theor. 2.*)

3. If a Number is divided into two Parts, the Product of the Whole and one Part, is equal to the Sum of the Square of this Part, and the Product of the Parts: This is also a Consequence, or particular Case of the 1st: Thus, if $N = a + b$, then $Na = a^2 + ab$, (*Euclid, Theor. 3.*)

4. If a Number is divided into two Parts, the Square of the Whole is equal to the Sum of the Squares of the Parts, and twice the Product of the Parts: This proceeds also from the 1st: Thus, if $N = a + b$, then $N^2 = a^2 + b^2 + 2ab$, (*Euclid, Theor. 4.*)

The rest of *Euclid* you have in the following *Theorems*.

THEOREM I.

THE Square of any Number is equal to the Difference of the Products of that Number, multiplied into any greater Number, and into the Difference of these Numbers. Or it is equal to the Sum of the Products of that Number multiplied into any lesser, and into the Difference of these Numbers.

DEMONSTR. (1) Let two Numbers be a , $a + d$, wherein d is the Difference; then $a \times \overline{a + d} = a^2 + ad$; hence $a^2 = a \times \overline{a + d} - ad$. (2.) Let the Numbers be $a - d$, a , wherein d is also the Difference; then $a \times \overline{a - d} = a^2 - ad$, and $a^2 = a \times \overline{a - d} + ad$.

THEO:

THEOREM II.

THE Sum of the Squares of two Numbers is equal to the Sum of twice their Product and the Square of their Difference.

DEMONSTR. Let the Roots be a, b , then $a^2 + b^2 = 2ab + \overline{a-b}^2$; for $\overline{a-b}^2 = a^2 - 2ab + b^2$; whence the Theorem is manifest.

Or let the Roots be $a, a+b$, wherein b is the Difference; then is $a^2 + \overline{a+b}^2 = a^2 + a^2 + 2ab + b^2 = 2a^2 + 2ab + b^2 = 2a \times \overline{a+b} + b^2$.

SCHOLIUM. In this last Form we see plainly comprehended *Euclid's Theor. 7.* which is this; If a Number consist of two Parts, (a, b), the Sum of the Squares of the Whole and of one Part, (*viz.* $\overline{a+b}^2 + a^2$) is equal to double the Product of the whole into this Part, and the Square of the other Part (*viz.* $2a \times \overline{a+b} + b^2$).

THEOREM III.

THE Sum of the Squares of two Numbers is equal to the $\frac{1}{2}$ Sum of the Squares of their Sum and Difference.

DEMONSTR. The two Numbers being a, b , then $a^2 + b^2 = \frac{\overline{a+b}^2 + \overline{a-b}^2}{2}$, for $\overline{a+b}^2 = a^2 + 2ab + b^2$, and $\overline{a-b}^2 = a^2 - 2ab + b^2$, and the Sum of these two Squares is $2a^2 + 2b^2$, whose $\frac{1}{2}$ is $a^2 + b^2$.

COROLL. The Sum of two Squares is double the Sum of the Squares of their half Sum and half Difference; for $a+b, a-b$, may represent any two Numbers, whose half Sum is a , and their half Difference is b ; but we see above that $2a^2 + 2b^2 = \overline{a+b}^2 + \overline{a-b}^2$.

SCHOLIUMS.

1. This Corollary is in effect the same as *Euclid, Theor. 9.* *viz.* If a Number is divided into two equal Parts, a, a , and into two unequal Parts $a+b, a-b$, (whose Sum is $2a$) the Sum of the Squares of the unequal Parts, (*viz.* $\overline{a+b}^2 + \overline{a-b}^2$) is equal to twice the Square of the half, (*viz.* $2 \times a^2$) and twice the Square of the middle Part, or half Difference of the unequal Parts (*viz.* $2 \times b^2$).

2. If we express the supposed Numbers thus, $a, a+b$, then the Theorem is $a^2 + \overline{a+b}^2 = \frac{2a^2 + \overline{a+b}^2 + b^2}{2}$. Also by taking b and $2a+b$ for two Numbers, whose Sum is $2a+b$, and their half Sum $a+b$, and half Difference a , the preceding Corollary is thus expressed, $2a^2 + \overline{a+b}^2 + b^2 = 2 \times \overline{a+b}^2 + 2 \times a^2$: Which is in effect the same as *Euclid, Theor. 10.* *viz.* If a Number is equally divided into two Parts, a, a , and to the whole $2a$ another Number be added as b , the Square of the Sum, *viz.* $\overline{2a+b}^2$, and the Square of the

the Number added, *viz.* b^2 , are together equal to double the Squares of $\frac{1}{2}$ the 1st Number, and of the Sum of that half and the Number added, *viz.* $2a^2 + 2 \times \overline{a+b}^2$.

THEOREM IV.

THE Sum of the Squares of two Numbers is equal to the Difference betwixt the Square of their Sum and double their Product.

DEMONSTR. The two Numbers being a, b , the Theorem is $a^2 + b^2 = \overline{a+b}^2 - 2ab$, for $\overline{a+b}^2 = a^2 + 2ab + b^2$, from which take $2ab$, remains $a^2 + b^2$.

Or thus; Let the two Numbers be $2a, b$, then $4a^2 + b^2 = \overline{2a+b}^2 - 4ab$.

COROLL. The Square of the Sum of two Numbers is 4 times their Product, more the Square of their Difference. This appears by making $a, a+b$, the two Numbers, whereby $2a+b$ is their Sum, and then adding $4ab$ to both Sides, whereby $\overline{2a+b}^2 = 4a^2 + 4ab + b^2 = 4 \times a \times \overline{a+b} + b^2$.

SCHOLIUM. This Coroll. is the same in effect as Euclid, Theor. 8. *viz.* If a Number is divided into two Parts, as $a, a+b$; then 4 times the Product of the whole, and one Part, more the Square of the other Part, is equal to the Squares of the Sum of the whole and that Part.

THEOREM V.

THE Square of the Sum of two Numbers is equal to the Sum of the Square of one of them; and the Product of the other into the Sum of this other and double the former.

Also the Square of the Difference of two Numbers is equal to the Difference of the Square of one of them, and the Product of the other into the Difference of this other, and double the former.

DEMONSTR. I. $\overline{a+b}^2 = a^2 + 2ab + b^2 = a^2 + \overline{2a+b} \times b$.

2. $\overline{a-b}^2 = a^2 - 2ab + b^2 = a^2 - \overline{2a-b} \times b$.

SCHOLIUM. The first Part of this comprehends Euclid, Theor. 6. *viz.* If a Number is divided into two equal Parts a, a , and to the whole $2a$, another Number b is added; the Product of the Sum $2a+b$ by the Number added b , *viz.* $\overline{2a+b} \times b$, &c. together with the Square of $\frac{1}{2}$ the first Number, *viz.* a^2 , is equal to the Square of the Sum of this half Number, and the Number added, *viz.* $\overline{a+b}^2$.

THEOREM VI.

THE Difference of the Squares of two Numbers is equal to the Product of their Sum and Difference.

DEMON. $\overline{a+b} \times \overline{a-b} = a^2 + ab - ab - b^2 = a^2 - b^2$.

COROLLARIES.

1. Of two unequal Numbers, $a+b, a-b$, the Square of half their Sum, *viz.* a^2 (for $2a$ is the Sum) is the Sum of their Product, *viz.* $a^2 - b^2$, and the Square of their Difference, *viz.* b^2 .

SCHOL.

SCHOLIUM. This is *Euclid's Theor. 5. viz.* If a Number is divided into two equal Parts, a, a ; and into two unequal Parts, $a+b, a-b$; the Product of the unequal Parts (*viz.* $a+b \times a-b = a^2 - b^2$) together with the Square of the middle Part, (*i. e.* of half their Difference, b^2) is equal to the Square of half the given Number, *viz.* a^2 .

2. The Sum and Difference of two Numbers are the reciprocal aliquot Parts of the Difference of their Squares.

3. The Square of any Number may be produced after a new Method. Thus: Let the given Root be N , assume any other Number A ; the Product of their Sum and Difference, which call p , is the Difference of their Squares, *i. e.* $p = N^2 - A^2$, or $A^2 - N^2$. Hence $N^2 = A^2 + p$, or $A^2 - p$.

4. Take any Number A , and make a Series from it continually decreasing by 1, till the least Term be 1; also a Series increasing by 1, to the same Number of Terms; multiply the corresponding Terms of the two Series together; the Products make a Series decreasing in such a manner, that comparing each Product to the Square of A , the Differences are the Series of Squares of the natural Progression, 1, 2, 3, &c. the Deduction of which from the *Theorem* is easy. Thus: $A - n \times A + n = A^2 - N^2$. But the Differences betwixt A ,

$A = 8. 7. 6. 5. 4. 3. 2. 1.$
 $A = 8. 9. 10. 11. 12. 13. 14. 15.$
 Products - - - 64. 63. 60. 55. 48. 39. 28. 15.
 Differs from 64. - - 1. 4. 9. 16. 25. 36. 49.

and the several Terms of the Series, are, by Supposition, 1, 2, 3, 4, &c. *that is*, n is gradually 1, 2, 3, &c. Consequently the Differences of the several Products of the corresponding Terms of the two Series from A^2 the first Product, are gradually the Squares of these Roots, 1, 2, 3, &c. Hence again,

5. We have this Rule for summing the Series of the Squares of the natural Progression 1, 2, 3, &c. *viz.* Take any Number A greater than n , the greatest of the Roots whose Squares are to be summed; then beginning at $A-1$, and $A+1$, continue a Series downwards from $A-1$, and upwards from $A+1$, with the common Difference 1, till the Number of Terms be n ; then taking the Products of the two Series as before, subtract

$A-1 : A-2 : A-3 : A-4, \&c.$
 $A+1 : A+2 : A+3 : A+4, \&c.$
 $A^2-1 : A^2-4 : A^2-9 : A^2-16, \&c.$

Sum of the Squares is $n \times A^2$ wanting the Sum of the Products.

their Sum from $n \times A^2$; the Remainder is the Sum sought. The Reason is plain; for the Sum of the Products is n times A^2 , wanting the Sum of the Series of Squares 1, 4, 9, &c. taken to a Number of Terms equal to n ; therefore also the

THEOREM VII.

The Sum of any Number of different Powers of the same Root, which stand all next together in the Series or Order of Powers, (*i. e.* whose Indexes follow one another in the natural Series of Numbers 1, 2, 3, &c. but beginning at any Power, or Place of the Series) is equal to the Quote of the Difference of the least of these Powers, and that next above the greatest of them, divided by the Difference of the Root and 1. Thus:

Example 1. $a^4 + a^5 + a^6 = \frac{a^7 - a^4}{a - 1}$

2. $a + a^2 + a^3 + a^4 = \frac{a^5 - a}{a - 1}$

Ex. 3. $a + a^2 + a^3, \&c. + a^n = \frac{a^{n+1} - a}{a - 1}$

4. $a^r + a^{r+1} + a^{r+2}, \&c. + a^{r+n} = \frac{a^{r+n+1} - a^r}{a - 1}$

$$\begin{array}{r} a + a^2 + a^3 + a^4, \&c. + a^n. \\ a - 1. \\ \hline a^2 + a^3 + a^4 + a^5, \&c. + a^{n+1}. \\ a + a^2 + a^3 + a^4, \&c. + a^n. \\ \hline a^{n+1} - a. \end{array}$$

DEMON. Take the Series $a + a^2 + a^3 + a^4$ to a^n , multiply it by $a-1$; the Product is $a^{n+1} - a$, as the annex'd Scheme of the Operation manifestly shews; For the given Series being multiplied by a , the Series of Products is the same as the given Series from the second Term, taking